

CURVES ON SURFACES, CHARTS, AND WORDS

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ABSTRACT. We give a combinatorial description of closed curves on oriented surfaces in terms of certain permutations, called charts. We describe automorphisms of curves in terms of charts and compute the total number of curves counted with appropriate weights. We also discuss relations between curves, words, and complex structures on surfaces.

1. INTRODUCTION

Words, defined as finite sequences of letters, were used to describe geometric figures already by the Greeks. It was common to label points by letters and to encode polygons by sequences of labels of their vertices, see [Eu]. Gauss [Ga] extended this method to closed curves with self-intersections. He considered generic curves on the plane that is smooth immersions $S^1 \rightarrow \mathbb{R}^2$ with only double transversal self-intersections. The Gauss word of a generic curve is obtained by labelling its self-crossings by different letters and writing down these letters in the order of their appearance when one traverses the curve. Each letter appears in this word twice. Since the work of Gauss, considerable efforts were directed at characterizing the words arising in this way from generic curves on \mathbb{R}^2 and on other surfaces, see [Ro], [RR], [DT], [CW], [CE], [CR].

The present paper is motivated by the desire to invert the procedure and to find a geometric presentation of arbitrary words by curves on (oriented) surfaces. A related question is to find a topological classification of curves on oriented surfaces. We consider in this paper all (closed) curves with finite number of self-crossings of any finite multiplicity ≥ 2 . We endow curves with a finite set of distinguished points of multiplicity 1, called corners. The set of corners is chosen arbitrary and may be empty. The Gauss word extends to such curves but is far from being a full invariant.

We introduce an invariant of curves called the “chart”. The chart of a curve f is a permutation of the set $\{-n, -n+1, \dots, -1, 1, \dots, n-1, n\}$ where $n \geq 1$ is an integer determined by f . The definition of the chart of a curve reminds of Grothendieck’s cartographic groups, cf. [JS].

Knowing the chart of a curve, we can recover the position of the curve in its narrow neighborhood in the ambient surface. For a *filling* curve, that is a curve on a closed oriented surface whose complementary regions are disks, the chart determines the curve and the surface up to homeomorphism. In particular, the genus of the surface can be computed directly from the chart; we give a combinatorial formula for this genus.

A curve $f : S^1 \rightarrow \Sigma$ may have automorphisms that is degree 1 homeomorphisms $\Sigma \rightarrow \Sigma$ preserving both the curve and its set of corners. The isotopy classes of such automorphisms form a group $\text{Aut}(f)$. If f is filling, then $\text{Aut}(f)$ is a finite cyclic group. We compute its order $|\text{Aut}(f)|$ in terms of the chart of f .

For self-transversal curves, i.e., curves with only transversal self-intersections (of arbitrary multiplicity), the information encoded in the chart can be presented in a more compact form of a semichart. A semichart is a pair (a permutation of the set $\{1, 2, \dots, n\}$, a subset of this set). Filling self-transversal curves are determined by their semicharts up to homeomorphism.

We use charts to compute the number of curves counted with certain weights. We state here a version of this result for filling self-transversal curves (on closed oriented surfaces of arbitrary genus). Let \mathcal{C}_{str} be the set of homeomorphism classes of such curves. Then we have an equality of formal power series

$$(1.0.1) \quad \sum_{f \in \mathcal{C}_{str}} \frac{1}{(n(f)-1)! |\text{Aut}(f)|} t_1^{k_1(f)} t_2^{k_2(f)} t_3^{k_3(f)} \dots = \prod_{m \geq 1} \exp \left(\frac{2^{m-1}}{m} t_m \right)$$

where t_1, t_2, \dots are independent variables, $k_1(f)$ is the number of corners of f , $k_m(f)$ with $m \geq 2$ is the number of crossings of f of multiplicity m , and $n(f) = \sum_{m \geq 1} m k_m(f)$. The monomials on the left hand side of (1.0.1) are finite since $k_m(f) = 0$ for any f and all sufficiently big m . Formula 1.0.1 indicates that in appropriate statistical terms, the crossings of different multiplicities are independent of each other.

Comparing the coefficients of the monomial $t_1^{k'} t_2^k$ with $k', k \geq 0$ on both sides of (1.0.1), we obtain

$$(1.0.2) \quad \sum_{f \in \mathcal{C}_{str}(k', k)} \frac{1}{|\text{Aut}(f)|} = \frac{(2k + k' - 1)!}{k! k'!}$$

where $\mathcal{C}_{str}(k', k) \subset \mathcal{C}_{str}$ is the set of homeomorphism classes of filling curves with k' corners, k double transversal crossings, and no other crossings. In particular, for $k' = 0$,

$$\sum_{f \in \mathcal{C}_{str}(0, k)} \frac{1}{|\text{Aut}(f)|} = \frac{(2k - 1)!}{k!}.$$

For $k' = 1$, Formula 1.0.2 gives $\text{card}(\mathcal{C}_{str}(1, k)) = 2k!/k!$ since $\text{Aut}(f) = 1$ for $f \in \mathcal{C}_{str}(1, k)$.

Although curves are purely topological objects, they are related to deep algebra and geometry. The image of a filling curve $f : S^1 \rightarrow \Sigma$ is a graph whose vertices are the self-intersections and the corners of f . In terminology of [Sc], p. 51, this graph is a pre-clean dessin d'enfants. By the classical Grothendieck construction, it induces on Σ a structure of an algebraic curve over $\overline{\mathbb{Q}}$. Over \mathbb{C} this gives a smooth curve with distinguished set of $k = \sum_{m \geq 1} k_m(f)$ points consisting of the self-intersections and the corners of f . Numerating them, we obtain a point of the moduli space $\mathcal{M}_{g, k}$ where $g = g(\Sigma)$ is the genus of Σ .

Another geometric construction applies when Σ is a smooth surface. It associates with a curve $f : S^1 \rightarrow \Sigma$ an oriented knot in the 3-manifold $S\Sigma$ formed by tangent vectors of Σ of length 1 with respect to a certain Riemannian metric on Σ . If f is smooth and self-transversal, this knot is formed by the unit positive tangent vectors of f . This knot is transversal to the standard contact structure on $S\Sigma$. For general f , we first approximate it by a smooth self-transversal curve and then proceed as above.

These geometric constructions can be combined with the construction of filling curves from charts and allow to associate with each chart an algebraic curve over $\overline{\mathbb{Q}}$, a point of a moduli space, and a knot in the tangent circle bundle over a surface.

Observe also that the charts of self-transversal curves can be naturally viewed as elements of Coxeter groups of type B .

Coming back to words, note that for curves on oriented surfaces, the letters in the associated word naturally acquire signs \pm . To simplify notation, we omit the minuses. Thus instead of writing $A^- B^+ A^+ C^-$ we write $AB^+ A^+ C$. We show that every such word (in any alphabet) can be realized by a curve on a surface. This realization is by no means unique. Not all words can be realized by self-transversal curves. We say that a word is odd if every letter appearing in this word (with or without superscript $+$) appears without $+$ an odd number of times. For instance, the words ABC , AA^+ , $A^+ B^+ AB$ are odd while AA , $A^+ A^+$, AB^+ are not odd. We prove that a word can be realized by a self-transversal curve if and only if it is odd.

A deeper connection between words and curves involves so-called coherent curves, generalizing the generic curves. Note that the branches of a self-transversal curve $f : S^1 \rightarrow \Sigma$ passing through a given crossing acquire a cyclic order obtained by moving on Σ around the crossing. We call f coherent if for all its crossings, this cyclic order coincides with the order of appearance of these branches when one traverses the curve. We prove that any odd word has a unique realization by a coherent filling (self-transversal) curve. This gives a geometric form to odd words: they correspond bijectively to the homeomorphism classes of coherent filling curves. For example, the word ABC corresponds to a triangle on S^2 with corners in the vertices. More generally, the word $A_1 A_2 \dots A_n$ formed by distinct letters A_1, A_2, \dots, A_n corresponds to an embedded n -gon on S^2 with corners in the vertices. The word AA^+ corresponds to a 8-like curve on S^2 . The word $A^+ B^+ AB$ corresponds to a curve on the torus. We formulate conditions on an odd word necessary and sufficient for the corresponding coherent curve to be planar. This extends a theorem of Rosenstiehl [Ro] on Gauss words.

Combining with the geometric constructions outlined above, we associate with each odd word an algebraic curve over $\overline{\mathbb{Q}}$, a point of a moduli space, and a knot in a 3-manifold.

We also introduce and study further geometric classes of curves (pointed/alternating/beaming/perfect curves). All these classes can be described in terms of their charts.

Throughout the paper, by a surface, we mean an *oriented* 2-dimensional manifold possibly with boundary.

2. CURVES AND HOMEOMORPHISMS

2.1. Tame curves. To avoid locally wild behavior, we consider only tame curves. A *tame curve* on a surface Σ is a continuous map f from $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ to $\Sigma - \partial\Sigma$ satisfying two conditions:

- (i) f is locally injective, i.e., each point of S^1 has a neighborhood U such that $f|_U : U \rightarrow \Sigma$ is injective;

(ii) for any $a \in f(S^1)$ there are a closed neighborhood $V \subset \Sigma$ and a homeomorphism of V onto the unit complex disk $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ sending a to 0 and sending $f(S^1) \cap V$ onto the set $\{z \in D \mid z^m \in \mathbb{R}\}$ where m is a positive integer (depending on a).

The integer $m = m_a$ in (ii) is the *multiplicity* of a . The set $\{z \in D \mid z^m \in \mathbb{R}\}$ consists of $2m$ radii of D . Moving along a radius towards $a = 0$ the curve then goes away along another radius. Clearly, $\text{card}(f^{-1}(a)) = m$.

A *crossing* of a tame curve $f : S^1 \rightarrow \Sigma$ is a point of $f(S^1)$ of multiplicity > 1 . The set of crossings of f is denoted $\text{cr}_{>1}(f)$. The compactness of S^1 implies that this set is finite. Observe that

$$f^{-1}(\text{cr}_{>1}(f)) = \{x \in S^1 \mid \text{there is } y \in S^1 - \{x\} \text{ such that } f(x) = f(y)\}.$$

If Σ is a smooth surface, then all smooth immersions $S^1 \rightarrow \Sigma$ with finite number of crossings are tame.

2.2. Corners and filling curves. A *tame curve with corners* on a surface Σ is a tame curve $f : S^1 \rightarrow \Sigma$ endowed with a finite (possibly empty) subset of $f(S^1) - \text{cr}_{>1}(f)$. This finite subset is denoted $\text{cr}_1(f)$, its points are called *corners* of f . They all have multiplicity 1. Set $\text{cr}(f) = \text{cr}_1(f) \cup \text{cr}_{>1}(f)$ and $\text{sing}(f) = f^{-1}(\text{cr}(f))$. Both these sets are finite. Set

$$n(f) = \text{card}(\text{sing}(f)) = \sum_{a \in \text{cr}(f)} m_a = \sum_{m \geq 1} m k_m(f)$$

where $k_1(f)$ is the number of corners of f and $k_m(f)$ with $m \geq 2$ is the number of crossings of f of multiplicity m . Clearly, $n(f) = 0$ if and only if f has no crossings and no corners.

From now on, the word “curve” means a tame curve with corners on a surface.

A curve $f : S^1 \rightarrow \Sigma$ is *filling* if Σ is a closed connected surface and all components of $\Sigma - f(S^1)$ are open 2-disks. Any curve f on any surface Σ gives rise to a filling curve by taking a regular closed neighborhood $U \subset \Sigma$ of $f(S^1)$ and gluing 2-disks to the components of ∂U .

2.3. Homeomorphisms. A *homeomorphism* of curves $f_1 : S^1 \rightarrow \Sigma_1, f_2 : S^1 \rightarrow \Sigma_2$ is a pair $(\varphi^{(1)} : S^1 \rightarrow S^1, \varphi^{(2)} : \Sigma_1 \rightarrow \Sigma_2)$ of orientation preserving homeomorphisms such that $f_2 \varphi^{(1)} = \varphi^{(2)} f_1$ and $\varphi^{(2)}(\text{cr}_1(f_1)) = \text{cr}_1(f_2)$. If there is such a pair, then f_1 and f_2 are *homeomorphic*. The role of $\varphi^{(1)}$ is to ensure that curves obtained from each other by re-parametrization are homeomorphic. Note that $\varphi^{(1)}$ maps $\text{sing}(f_1)$ bijectively onto $\text{sing}(f_2)$ and $\varphi^{(2)}$ maps $f_1(S^1), \text{cr}_{>1}(f_1), \text{cr}_1(f_1)$ bijectively onto $f_2(S^1), \text{cr}_{>1}(f_2), \text{cr}_1(f_2)$, respectively.

A curve homeomorphic to a filling curve is itself filling. The main problem of the combinatorial theory of curves is a classification of filling curves up to homeomorphism.

2.4. Examples. 1. An embedding $S^1 \hookrightarrow S^2$ with empty set of corners is a *trivial* curve. Any two trivial curves are homeomorphic.

2. Let $A, B, C, D \in \mathbb{R}^2 \subset S^2$ be consecutive vertices of a square with center O where \mathbb{R}^2 is oriented so that the pair of vectors (AB, AC) is positive. The closed broken lines $ABODCOA$ and $ABOCDOA$ with no corners are filling curves on S^2 with one crossing O of multiplicity 2. They are not homeomorphic.

3. Let $A, B, C, D, E, F \in \mathbb{R}^2 \subset S^2$ be consecutive vertices of a regular hexagon with center O where the plane is oriented as above. The closed broken line $ABOEFODCOA$ with no corners is a filling curve on S^2 (called *trifolium*) with one crossing O of multiplicity 3.

3. CHARTS AND FLAGS

3.1. Preliminaries. For a positive integer n , set $\hat{n} = \{1, 2, \dots, n\}$ and

$$\bar{n} = (-\hat{n}) \cup \hat{n} = \{-n, -n+1, \dots, -2, -1, 1, 2, \dots, n-1, n\}.$$

The *circular permutation* $\sigma_n : \bar{n} \rightarrow \bar{n}$ sends $\pm k$ to $\pm(k+1)$ for $k = 1, 2, \dots, n-1$ and sends $\pm n$ to ± 1 . For $n = 0$, set $\hat{n} = \emptyset$, $\bar{n} = \{0\}$, and $\sigma_0 = \text{id} : \{0\} \rightarrow \{0\}$.

For a finite set F and a bijection $t : F \rightarrow F$, an *orbit* of t , or shorter a *t-orbit*, is a minimal non-empty t -invariant subset of F . The unique orbit of t containing a given element $a \in F$ consists of m elements $a, t(a), t^2(a), \dots, t^{m-1}(a)$ where m is the minimal positive integer such that $t^m(a) = a$. The mapping t determines a cyclic order \prec on this orbit by $a \prec t(a) \prec t^2(a) \prec \dots \prec t^{m-1}(a) \prec a$. It is clear that F is a disjoint union of orbits of t . The set of orbits of t is denoted F/t .

3.2. Charts. A pair (n, t) consisting of an integer $n \geq 0$ and a bijection $t : \bar{n} \rightarrow \bar{n}$ is a *chart* if for any $k \in \bar{n}$, there is $s \in \mathbb{Z}$ such that $t^s(k) = -k$. The chart $(0, \text{id} : \{0\} \rightarrow \{0\})$ is called the *trivial* chart. In a non-trivial chart (n, t) the bijection t is necessarily fixed-point free: if $t(k) = k$, then $t^s(k) = k \neq -k$ for $s \in \mathbb{Z}$. Any orbit of t is invariant under the negation $\bar{n} \rightarrow \bar{n}, k \mapsto -k$ and has an even number of elements. The cyclic group $\mathbb{Z}/n\mathbb{Z}$ acts on the set of charts (n, t) via $t \mapsto \sigma_n t (\sigma_n)^{-1}$.

For each curve f , we shall define a chart $(n(f), t(f))$ where $n(f)$ is the number defined in Section 2.2. The following theorem gives a topological classification of filling curves in terms of charts.

Theorem 3.2.1. *The formula $f \mapsto (n(f), t(f))$ defines a bijective correspondence between filling curves considered up to homeomorphism and charts considered up to conjugation by the circular permutation.*

Theorem 3.2.1 directly follows from Lemmas 3.4.1 and 3.5.1 stated below. This theorem provides a combinatorial method of presenting filling curves: to specify a curve, it suffices to specify its chart.

Note that all curves on S^2 are filling. Therefore the chart is a full topological invariant of curves on S^2 .

3.3. Flags. A *flag* of a curve f is a pair $(x \in \text{sing}(f), \varepsilon = \pm)$. The flag $(x, -)$ is *incoming*, the flag $(x, +)$ is *outgoing*. The flags $(x, -)$ and $(x, +)$ are said to be *opposite*. With a flag $r = (x, \varepsilon)$ we associate a small arc in S^1 beginning at x and going counterclockwise if $\varepsilon = +$ and clockwise if $\varepsilon = -$. The image of this arc under f is a small embedded arc on $f(S^1)$ with one endpoint at $f(x)$. The latter arc presents r geometrically. We say that r is a *flag at* $f(x)$ and $f(x)$ is the *root of* r . Each $a \in \text{cr}(f)$ is a root of $2m_a$ flags.

The set of flags of f is denoted $\text{Fl}(f)$. This set has $2n$ elements where $n = n(f) = \text{card}(\text{sing}(f))$. If $n \neq 0$, then $\text{Fl}(f)$ can be identified with the set \bar{n} as follows. Starting at a point of $S^1 - \text{sing}(f)$ and traversing S^1 counterclockwise denote the points of $\text{sing}(f)$ consecutively x_1, x_2, \dots, x_n . The flag (x_k, \pm) corresponds to $\pm k \in \bar{n}$ for $k = 1, 2, \dots, n$. The resulting bijection $\text{Fl}(f) \approx \bar{n}$ is well defined up to composing with σ_n .

3.4. Charts of curves. For a curve $f : S^1 \rightarrow \Sigma$, we define a bijective map $t : \text{Fl}(f) \rightarrow \text{Fl}(f)$ called *flag rotation*. Starting from a flag of f at $a \in \text{cr}(f)$ and circularly moving on Σ in the positive direction around a , we numerate the flags of f at a by the residues $1, 2, \dots, 2m_a - 1, 2m_a \pmod{2m_a}$. The map t transforms the k -th flag at a into the $(k+1)$ -st flag at a . In particular, if $a \in \text{cr}_1(f)$, then t permutes the two (opposite) flags at a .

Assigning to each flag its root, we obtain $\text{Fl}(f)/t = \text{cr}(f)$. The orbit of t corresponding to $a \in \text{cr}(f)$ consists of the $2m_a$ flags of f at a . Opposite flags always lie in the same orbit.

If $n(f) = 0$, then by definition the chart of f is the trivial chart $(0, \text{id} : \bar{0} \rightarrow \bar{0})$. Suppose that $n = n(f) \neq 0$. Conjugating $t : \text{Fl}(f) \rightarrow \text{Fl}(f)$ by the bijection $\text{Fl}(f) \approx \bar{n}$, we obtain a bijection $\bar{n} \rightarrow \bar{n}$ also denoted t or $t(f)$. It is determined by f up to conjugation by σ_n . The pair $(n, t : \bar{n} \rightarrow \bar{n})$ is a chart called the *chart of* f . Note the identifications $\bar{n}/t = \text{Fl}(f)/t = \text{cr}(f)$. The opposite choice of orientation on Σ yields (n, t^{-1}) . Since the chart of f is entirely defined inside a narrow neighborhood of $f(S^1) \subset \Sigma$, it depends only on the filling curve determined by f .

Lemma 3.4.1. *Any chart (n, t) is the chart of a filling curve.*

Proof. It is enough to consider the case $n \geq 1$. Set $x_k = \exp(2k\pi i/n) \in S^1$ for $k \in \hat{n} = \{1, 2, \dots, n\}$. Starting in x_k and moving along S^1 counterclockwise (resp. clockwise) to the distance $\pi/2n$ we sweep an arc on S^1 denoted α_k (resp. α_{-k}). The arcs $\{\alpha_k, \alpha_{-k}\}_{k \in \hat{n}}$ are disjoint except that $\alpha_k \cap \alpha_{-k} = \{x_k\}$ for all k .

Identifying x_k and x_j (with $k, j \in \hat{n}$) each time there is a power of t transforming k into j , we obtain from S^1 a 1-dimensional CW-complex Γ . The 0-cells (vertices) of Γ are the points $p(x_1), p(x_2), \dots, p(x_n)$ where $p : S^1 \rightarrow \Gamma$ is the natural projection. All other points of Γ form the open 1-cells.

We shall thicken Γ to a surface. First we thicken each vertex $v \in \Gamma$ to a disk as follows. Set $J_+ = \{j \in \hat{n} \mid p(x_j) = v\}$ and $J = (-J_+) \cup J_+ \subset \bar{n}$. We claim that J is an orbit of t . To see this, pick any $k \in J_+$ and denote its t -orbit by $[k]$. We show that $J = [k]$. By the definition of Γ , we have $J_+ \subset [k]$. Hence, by the definition of a chart, $-J_+ \subset [k]$ and $J \subset [k]$. To prove the inclusion $[k] \subset J$, we show that $t^s k \in J$ for any $s \in \mathbb{Z}$. Pick $s_- \in \mathbb{Z}$ such that $-t^{s_-} k = t^{s_-} k$. If $t^{s_-} k \in \hat{n}$, then $t^{s_-} k \in J_+ \subset J$ by the definition of Γ . If $t^{s_-} k \in -\hat{n}$, then $-t^{s_-} k = t^{s_-} k \in J_+$ and $t^{s_-} k \in -J_+ \subset J$.

Set $m = \text{card}(J_+)$. The set $V_v = \cup_{r \in J} \alpha_r$ is a neighborhood of v in Γ consisting of $2m$ embedded arcs $\{\alpha_r\}_{r \in J}$ meeting at v . We embed V_v in a copy $D(v)$ of the unit 2-disk $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ as follows. Pick $r \in J$. The embedding $V_v \rightarrow D(v)$ sends v to 0 and sends the arc $\alpha_{t^s r}$ onto the radius $R_s = \exp(s\pi i/m) \cdot [0, 1]$ of $D(v)$ for $s = 0, 1, \dots, 2m-1$. In this way v is thickened to a copy $D(v)$ of D . We endow $D(v)$ with

counterclockwise orientation. Note that going around v in the positive direction we cross the arcs $\{\alpha_r\}_{r \in J}$ in the cyclic order determined by t as in Section 3.1.

Endow a 1-cell γ of Γ with the orientation induced by the counterclockwise orientation on S^1 . Then γ leads from a vertex v_1 of Γ to a vertex v_2 of Γ (possibly $v_1 = v_2$). These vertices are thickened above to disks $D_1 = D(v_1), D_2 = D(v_2)$. For $l = 1, 2$, the disk D_l meets γ along a radius $R_{s(l)}$ as above with endpoint $a_l \in \partial D_l = S^1$. The part of γ not lying in $\text{Int } D_1 \cup \text{Int } D_2$ is a closed interval $\gamma_\bullet \subset \gamma$ with endpoints a_1, a_2 . We thicken γ_\bullet to a ribbon $\gamma_\bullet \times [-1, 1]$ glued to $D_1 \cup D_2$ along the embedding $a_1 \times [-1, 1] \hookrightarrow \partial D_1$ sending $(a_1, u \in [-1, 1])$ to $\exp(u\pi i/2n) \cdot a_1$ and the embedding $a_2 \times [-1, 1] \hookrightarrow \partial D_2$ sending (a_2, u) to $\exp(-u\pi i/2n) \cdot a_2$. Note that the orientation on D_1, D_2 extends to their union with the ribbon.

Thickening in this way all cells of Γ we embed Γ into a surface U . By construction, U is a compact connected oriented surface with non-void boundary. All components of $U - \Gamma$ are homeomorphic to $S^1 \times [0, 1]$. Gluing 2-disks to the components of ∂U we obtain a closed connected oriented surface $\Sigma \supset U \supset \Gamma$ such that all components of $\Sigma - \Gamma$ are open disks. Composing the projection $p : S^1 \rightarrow \Gamma$ with the inclusion $\Gamma \hookrightarrow \Sigma$ we obtain a filling curve $S^1 \rightarrow \Sigma$. We choose its set of corners to be $\{p(x_k) \mid k \in \hat{n}, t(k) = -k\}$. It is clear from the construction that the chart of the resulting curve is the original chart (n, t) . \square

Remark 3.4.2. The construction of Σ is well known in the context of generic curves, see [Fr], [Ca], [CW].

3.5. Action of homeomorphisms. A homeomorphism $\varphi = (\varphi^{(1)} : S^1 \rightarrow S^1, \varphi^{(2)} : \Sigma_1 \rightarrow \Sigma_2)$ of curves $f_1 : S^1 \rightarrow \Sigma_1, f_2 : S^1 \rightarrow \Sigma_2$ induces a bijection $\varphi_* : \text{Fl}(f_1) \rightarrow \text{Fl}(f_2)$. It sends the flag of f_1 represented by a small embedded arc $\alpha \subset f_1(S^1)$ with root $a \in \text{cr}(f_1)$ to the flag of f_2 represented by the arc $\varphi^{(2)}(\alpha) \subset f_2(S^1)$ with root $\varphi^{(2)}(a) \in \text{cr}(f_2)$. The same φ_* is defined by $\varphi_*((x, \pm)) = (\varphi^{(1)}(x), \pm)$ where $x \in \text{sing}(f_1)$. The bijection φ_* sends opposite flags to opposite flags and commutes with the flag rotation. This implies that homeomorphic curves have the same charts. The next lemma establishes the converse for filling curves.

Lemma 3.5.1. *Filling curves having the same charts are homeomorphic.*

Proof. If one of the curves is trivial, then the claim is obvious. Consider the case of non-trivial curves. The reconstruction of a curve from its chart given in the proof of Lemma 3.4.1 is canonical except at the place where we pick $r \in J$. A different choice of r would lead to another embedding $V_v \hookrightarrow D$ obtained from the first one by composing with a rotation of D around its center to an angle proportional to $2\pi/m$. This, however, gives the same thickening of v to a disk $D(v)$; only the identification of $D(v)$ with D differs by this rotation. Thus, knowing the chart of a curve $f : S^1 \rightarrow \Sigma$ we can reconstruct a regular neighborhood $U \subset \Sigma$ of $f(S^1)$ and the curve $f : S^1 \rightarrow U$ up to homeomorphism. Since a homeomorphism of circles extends to a homeomorphism of disks bounded by these circles (cf. Lemma 4.2.2 (i) below), we conclude that knowing the chart of a filling curve we can reconstruct the curve up to homeomorphism. \square

3.6. Examples. 1. The chart of an embedding $S^1 \hookrightarrow S^2$ with n corners is the negation $\bar{n} \rightarrow \bar{n}, k \mapsto -k$.

2. The chart of the curve $ABOCDOA$ (Example 2.4.2) is the cyclic permutation $(1, -2, 2, -1)$ of $\bar{2}$ sending 1 to -2 , -2 to 2, 2 to -1 , and -1 to 1. The chart of the 8-like curve $ABODCOA$ (the same example) is the cyclic permutation $(1, 2, -1, -2)$ of $\bar{2}$. The chart of the same curve $ABODCOA$ with corners A, B, C, D is the permutation $(1, -1)(2, -2)(3, 6, -3, -6)(4, -4)(5, -5)$ of $\bar{6}$.

3. The chart of the curve $ABOEFODCOA$ (Example 2.4.3) is the cyclic permutation $(1, -2, 3, -1, 2, -3)$ of $\bar{3}$.

3.7. Riemann surfaces. By the results above, any chart (n, t) determines a filling curve $f : S^1 \rightarrow \Sigma$ uniquely up to homeomorphism. Applying to the graph $f(S^1)$ the Grothendieck construction, we obtain an algebraic curve over $\bar{\mathbb{Q}}$ and a point of the moduli space $\mathcal{M}_{g,k}$ where $g = g(\Sigma)$ and $k = \text{card}(\text{cr}(f)) = \text{card}(\bar{n}/t)$. Here to fix an order on the set $\text{cr}(f)$ we identify it with the set of orbits of $t : \bar{n} \rightarrow \bar{n}$ and order the latter by $A < B$ if $\min_{a \in A} |a| < \min_{b \in B} |b|$. Note that under conjugation of t by σ_n , this order may change.

These constructions can be generalized in various directions. Having positive real numbers r_1, \dots, r_n we can provide the graph $f(S^1)$ with a metric assigning r_i to the only edge containing the i -th outgoing flag for $i = 1, \dots, n$. The resulting metric graph determines a complex structure on Σ , see for instance [MP]. For $r_1 = \dots = r_n = 1$, this gives the same Riemann surface as in the Grothendieck construction on $f(S^1)$.

The construction in Lemma 3.4.1 can be applied to an arbitrary bijection $T : \bar{n} \rightarrow \bar{n}$, not necessarily a chart. The difference is that a vertex $v \in \Gamma$ is now thickened to a union of several disks meeting each other at the center; the number of these disks is equal to the number of T -orbits contained in the set $J = J(v)$. This yields a curve with corners on a *singular surface* $\Sigma^s = \Sigma^s(T)$ obtained from a closed (oriented, possibly

non-connected) surface $\Sigma = \Sigma(T)$ by contracting several disjoint finite subsets. The image of this curve a pre-clean dessin d'enfants on Σ which makes each component of Σ an algebraic curve over $\overline{\mathbb{Q}}$ and yields a point of the corresponding moduli space. Consider in more detail the case where all singularities of Σ^s are *simple* in the sense that Σ^s is obtained from Σ by contracting disjoint pairs of points. It happens iff for any T -orbit $A \subset \overline{n}$ there is a T -orbit $A' \subset \overline{n}$ such that $A \cup A'$ is invariant under negation on \overline{n} . Under this condition, T gives rise to a modular graph (τ, g) in the sense of [Ma], p. 88. Its vertices are numerated by the components of Σ ; its tails are numerated by negation invariant T -orbits; its edges are numerated by unordered pairs (A, A') of T -orbits such that A, A' are not negation invariant but their union is. The function g assigns to each vertex of τ the genus of the corresponding component of Σ . In terminology of [Ma], Σ^s is the combinatorial type of the “prestable labeled curve” represented by (τ, g) . If the modular graph (τ, g) is stable, then our constructions give a point of the Mumford-Deligne compactification $\overline{\mathcal{M}}_{G,k}$ where k is the number of tails of (τ, g) and G is the sum of all values of g plus the first Betti number of the 1-dimensional CW-complex underlying the graph τ .

4. AUTOMORPHISMS

4.1. Automorphisms of charts. An *automorphism* of a chart (n, t) is a permutation $\varphi : \overline{n} \rightarrow \overline{n}$ such that $\varphi t = t\varphi$ and φ is a power of the circular permutation $\sigma_n : \overline{n} \rightarrow \overline{n}$. The automorphisms of (n, t) form a group with respect to composition, it is denoted $\text{Aut}(t)$. This group is a subgroup of the cyclic group of order n generated by σ_n . Therefore $\text{Aut}(t)$ is a cyclic group of order dividing n . Clearly, $\text{Aut}(t) = \text{Aut}(\sigma_n t(\sigma_n)^{-1})$.

For example, consider the charts $t_1, t_2, t_3 : \overline{4} \rightarrow \overline{4}$ given as products of two cycles

$$t_1 = (1, 3, -3, -1)(2, 4, -4, -2), \quad t_2 = (1, -1, 3, -3)(2, -4, 4, -2), \quad t_3 = (1, -1, 3, -3)(2, -2, 4, -4).$$

It is easy to check that $\text{Aut}(t_1) = 1$, $\text{Aut}(t_2) = \mathbb{Z}/2\mathbb{Z}$, and $\text{Aut}(t_3) = \mathbb{Z}/4\mathbb{Z}$.

Lemma 4.1.1. *For any chart (n, t) and $m \geq 1$, the order of $\text{Aut}(t)$ divides $m k_m$ where k_m is the number of orbits of t consisting of $2m$ elements.*

Proof. Let $\overline{M} \subset \overline{n}$ be the union of all orbits of t consisting of $2m$ elements. Set $M = \overline{M} \cap \hat{n}$ where $\hat{n} = \{1, 2, \dots, n\}$. By the definition of a chart, $\overline{M} = M \cup (-M)$. Any automorphism $\varphi : \overline{n} \rightarrow \overline{n}$ of t fixes \overline{M} setwise. Since φ is a power of σ_n , it fixes \hat{n} setwise. Therefore $\varphi(M) = M$. The mapping $\varphi|_M : M \rightarrow M$ preserves the cyclic order on M induced by the standard cyclic order $1 \prec 2 \prec \dots \prec n \prec 1$ on \hat{n} . Therefore $(\varphi|_M)^{\text{card}(M)} = \text{id}$. A power of σ_n having a fixed point is the identity. Hence $\varphi^{\text{card}(M)} = \text{id}$. Taking as φ a generator of the cyclic group $\text{Aut}(t)$, we obtain that $|\text{Aut}(t)|$ divides $\text{card}(M) = (1/2) \text{card}(\overline{M}) = m k_m$. \square

4.2. Automorphisms of curves. An *automorphism* of a curve $f : S^1 \rightarrow \Sigma$ is a homeomorphism of f onto itself, that is a pair $\varphi = (\varphi^{(1)} : S^1 \rightarrow S^1, \varphi^{(2)} : \Sigma \rightarrow \Sigma)$ of orientation preserving homeomorphisms such that $f\varphi^{(1)} = \varphi^{(2)}f$ and $\varphi^{(2)}(\text{cr}_1(f)) = \text{cr}_1(f)$. These conditions imply that $\varphi^{(2)}$ preserves $f(S^1)$, $\text{cr}_{>1}(f)$, and $\text{cr}_1(f)$ setwise. Automorphisms $\varphi = (\varphi^{(1)}, \varphi^{(2)})$ and $\psi = (\psi^{(1)}, \psi^{(2)})$ of f can be multiplied by $\varphi\psi = (\varphi^{(1)}\psi^{(1)}, \varphi^{(2)}\psi^{(2)})$. With this multiplication, the automorphisms of f form a group. Two automorphisms of f are *isotopic* if they can be included in a continuous 1-parameter family of automorphisms of f . Quotienting the group of automorphisms of f by isotopy we obtain a group $\text{Aut}(f)$ whose elements are isotopy classes of automorphisms of f . The following theorem computes $\text{Aut}(f)$ in terms of the chart of f .

Theorem 4.2.1. *For any filling curve f with chart (n, t) , we have $\text{Aut}(f) = \text{Aut}(t)$.*

Proof. We begin with a simple and well known geometric lemma. The second claim of this lemma is due to J. W. Alexander and H. Tietze.

Lemma 4.2.2. *Let $D = D^N$ be a closed N -dimensional ball. Then*

- (i) *any homeomorphism $g : \partial D \rightarrow \partial D$ extends to a homeomorphism $\tilde{g} : D \rightarrow D$ such that under continuous deformation of g its extension \tilde{g} deforms continuously and for $g = \text{id}_{\partial D}$ we have $\tilde{g} = \text{id}_D$;*
- (ii) *any homeomorphism $D \rightarrow D$ fixing ∂D pointwise is isotopic to the identity in the class of homeomorphisms $D \rightarrow D$ fixing ∂D pointwise.*

Proof. We identify D with the unit Euclidean ball $\{z \in \mathbb{R}^N, \|z\| \leq 1\}$ where $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^N . Then $\partial D = \{z \in \mathbb{R}^N, \|z\| = 1\}$. For a homeomorphism $g : \partial D \rightarrow \partial D$ and $z \in D - \{0\}$, set $\tilde{g}(z) = \|z\| g(z/\|z\|)$. Set $\tilde{g}(0) = 0$. This defines a homeomorphism $\tilde{g} : D \rightarrow D$ extending g and satisfying (i).

For a homeomorphism $h : D \rightarrow D$ fixing ∂D pointwise, the formula

$$h_s(z) = \begin{cases} z, & \text{if } s \leq \|z\| \leq 1, \\ s h(z/s), & \text{if } \|z\| < s \end{cases}$$

defines an isotopy (= a continuous family of homeomorphisms) $\{h_s : D \rightarrow D\}_{s \in [0,1]}$ of $h_0 = \text{id}$ to $h_1 = h$. \square

Let f be a filling curve with chart (n, t) . If $n = 0$, then f is a trivial curve. Using Lemma 4.2.2 (ii) and the fact that orientation preserving homeomorphisms $S^1 \rightarrow S^1$ are isotopic to the identity, we obtain $\text{Aut}(f) = 1 = \text{Aut}(t)$. Assume from now on that $n > 0$. As we know, any automorphism φ of f induces a permutation $\varphi_* : \text{Fl}(f) \rightarrow \text{Fl}(f)$. Isotopic automorphisms of f induce the same permutations. (Indeed, a continuous deformation of a permutation of $\text{Fl}(f)$ must be constant since $\text{Fl}(f)$ is finite.) Therefore the formula $F(\varphi) = \varphi_*$ defines a group homomorphism $F : \text{Aut}(f) \rightarrow \text{Iso}(\text{Fl}(f))$ where for a set S , we denote by $\text{Iso}(S)$ the group of bijections $S \rightarrow S$ with multiplication given by composition.

Lemma 4.2.3. *The homomorphism $F : \text{Aut}(f) \rightarrow \text{Iso}(\text{Fl}(f))$ is injective.*

Proof. Let $\varphi = (\varphi^{(1)}, \varphi^{(2)})$ be an automorphism of f whose isotopy class lies in the kernel of F . We shall show that φ is isotopic to the identity. We prove first that φ is isotopic to an automorphism $\eta = (\eta^{(1)}, \eta^{(2)})$ of f with $\eta^{(1)} = \text{id}_{S^1}$. Since $\varphi_* = F(\varphi)$ acts as the identity on all flags of f , the homeomorphism $\varphi^{(1)} : S^1 \rightarrow S^1$ must fix $\text{sing}(f)$ pointwise. The points of $\text{sing}(f)$ split S^1 into $n = n(f)$ consecutive arcs $\gamma_1, \dots, \gamma_n$. Since $\varphi^{(1)}$ is orientation preserving and constant on $\partial\gamma_k$, it maps γ_k onto itself for all k . By Lemma 4.2.2 (ii), the restriction of $\varphi^{(1)}$ to γ_k is isotopic to the identity in the class of homeomorphisms $\gamma_k \rightarrow \gamma_k$ fixing $\partial\gamma_k$ pointwise. Gluing these isotopies over $k = 1, \dots, n$, we obtain a continuous family of homeomorphisms $\{\varphi_s^{(1)} : S^1 \rightarrow S^1\}_{s \in [0,1]}$ fixing $\text{sing}(f)$ pointwise and such that $\varphi_0^{(1)} = \varphi^{(1)}, \varphi_1^{(1)} = \text{id}$. The graph $f(S^1)$ is obtained from S^1 by identifying the points of $\text{sing}(f)$ having the same image under f . Since the homeomorphism $\varphi_s^{(1)} : S^1 \rightarrow S^1$ fixes $\text{sing}(f)$ pointwise, it induces a homeomorphism $\psi_s : f(S^1) \rightarrow f(S^1)$ such that $\psi_s f = f \varphi_s^{(1)}$, ψ_s fixes $\text{cr}(f)$ pointwise, and $\psi_1 = \text{id}$. Since f is a filling curve, the splitting of Σ along $f(S^1)$ gives a finite family of closed 2-disks $\{D_q\}_q$. Each ψ_s induces a homeomorphism $\partial D_q \rightarrow \partial D_q$. By Lemma 4.2.2 (i), the latter extends to a homeomorphism $D_q \rightarrow D_q$. Gluing these homeomorphisms along $f(S^1)$, we obtain a continuous family of homeomorphisms $\{\tilde{\psi}_s : \Sigma \rightarrow \Sigma\}_{s \in [0,1]}$ such that $\tilde{\psi}_s|_{f(S^1)} = \psi_s$ for all $s \in [0,1]$ and $\tilde{\psi}_1 = \text{id}_\Sigma$. Then $\tilde{\psi}_s f = \psi_s f = f \varphi_s^{(1)} : S^1 \rightarrow \Sigma$. Thus we obtained a continuous family $(\varphi_s^{(1)} : S^1 \rightarrow S^1, \tilde{\psi}_s : \Sigma \rightarrow \Sigma)$ of automorphisms of f such that $\varphi_0^{(1)} = \varphi^{(1)}$ and $(\varphi_1^{(1)}, \tilde{\psi}_1) = \text{id}$. The family $\{\eta_s = (\varphi_s^{(1)}(\varphi_s^{(1)})^{-1}, \varphi_s^{(2)}(\tilde{\psi}_s)^{-1})\}_{s \in [0,1]}$ of automorphisms of f is an isotopy between $\eta_0 = (\text{id}_{S^1}, \varphi^{(2)}(\tilde{\psi}_0)^{-1})$ and $\eta_1 = \varphi$. Thus φ is isotopic to an automorphism $\eta = \eta_0$ of f with $\eta^{(1)} = \text{id}_{S^1}$.

We prove now that any such η is isotopic to the identity. The equality $\eta^{(2)} f = f \eta^{(1)} = f$ shows that $\eta^{(2)} : \Sigma \rightarrow \Sigma$ preserves $f(S^1)$ pointwise. Splitting Σ along $f(S^1)$ we obtain a finite family of closed 2-disks $\{D_q\}_q$. Since $\eta^{(2)}$ preserves orientation, it has to map each D_q into itself fixing ∂D_q pointwise. By Lemma 4.2.2 (ii), the resulting self-homeomorphism of D_q is isotopic to the identity in the class of self-homeomorphism of D_q fixing ∂D_q pointwise. Gluing such isotopies over all q , we obtain an isotopy of $\eta^{(2)}$ to the identity in the class of homeomorphisms $\Sigma \rightarrow \Sigma$ fixing $f(S^1)$ pointwise. All these homeomorphisms are automorphisms of f acting on S^1 as the identity. Hence η is isotopic to the identity automorphism of f . \square

Lemma 4.2.4. *Under the identification $\text{Fl}(f) = \bar{n}$, the image of $F : \text{Aut}(f) \rightarrow \text{Iso}(\text{Fl}(f)) = \text{Iso}(\bar{n})$ is $\text{Aut}(t)$.*

Proof. Pick $\varphi \in \text{Aut}(f)$. By Section 3.5, the bijection $F(\varphi) = \varphi_* : \text{Fl}(f) \rightarrow \text{Fl}(f)$ commutes with the flag rotation. This bijection is defined by $\varphi_*((x, \pm)) = (\varphi^{(1)}(x), \pm)$ where $x \in \text{sing}(f)$. The homeomorphism $\varphi^{(1)} : S^1 \rightarrow S^1$ sends $\text{sing}(f)$ onto itself preserving the cyclic order. Hence under the identification $\text{Fl}(f) = \bar{n}$, the bijection φ_* becomes a power of the circular permutation $\sigma = \sigma_n : \bar{n} \rightarrow \bar{n}$. Thus, $\varphi_* \in \text{Aut}(t)$.

It remains to prove that all the powers of σ commuting with t lie in the image of F . Note that if this holds for a curve homeomorphic to f , then this holds for f . Therefore it is enough to prove our claim in the case where f is the filling curve constructed from the chart (n, t) in the proof of Lemma 3.4.1.

We use notation introduced there: the points $x_k = \exp(2k\pi i/n) \in S^1$, the graph Γ , and the projection $p : S^1 \rightarrow \Gamma$. Let $\tau = \sigma^q$ with $q \in \mathbb{Z}$ be a power of σ commuting with t . Let $\varphi^{(1)} : S^1 \rightarrow S^1$ be multiplication by $\exp(2q\pi i/n)$. Clearly, $\varphi^{(1)}(x_k) = x_{\tau(k)}$ for all k . If $p(x_k) = p(x_j)$ with $k, j \in \hat{n}$, then there is a power of t transforming j into k . Since $t\tau = \tau t$, the same power of t transforms $\tau(j)$ into $\tau(k)$. Hence $p\varphi^{(1)}(x_k) =$

$p(x_{\tau(k)}) = p(x_{\tau(j)}) = p\varphi^{(1)}(x_j)$. Therefore $\varphi^{(1)}$ induces a map $\psi : \Gamma \rightarrow \Gamma$ such that $\psi p = p\varphi^{(1)}$. Applying the same argument to the inverses of τ and $\varphi^{(1)}$, we obtain an inverse map. Hence ψ is a homeomorphism.

For a vertex $v = p(x_k) \in \Gamma$, the point $\psi(v) = p\varphi^{(1)}(x_k) = p(x_{\tau(k)})$ is also a vertex of Γ . Clearly, ψ maps the neighborhood $V_v \subset \Gamma$ of v defined in Lemma 3.4.1 onto $V_{\psi(v)}$. The homeomorphism $\psi|_{V_v} : V_v \rightarrow V_{\psi(v)}$ transforms a flag (x_k, \pm) at v into the flag $(x_{\tau(k)}, \pm)$ at $\psi(v)$. Since $t\tau = \tau t$, this transformation of flags is t -equivariant. Therefore $\psi|_{V_v}$ maps the arcs forming V_v onto the arcs forming $V_{\psi(v)}$ preserving their cyclic order induced by t . This implies that $\psi|_{V_v}$ extends to an orientation preserving homeomorphism $D(v) \rightarrow D(\psi(v))$. The resulting self-homeomorphism of $\Gamma \cup \cup_v D(v)$ obviously extends to the ribbons used in Lemma 3.4.1 to construct U . This gives an orientation preserving homeomorphism $U \rightarrow U$. By Lemma 4.2.2 (i), the latter extends to a homeomorphism $\varphi^{(2)} : \Sigma \rightarrow \Sigma$. Since $\varphi^{(2)}$ is an extension of ψ , we have $i\psi = \varphi^{(2)}i$ where i is the inclusion $\Gamma \hookrightarrow \Sigma$. Recall that $f = ip : S^1 \rightarrow \Sigma$. Therefore $f\varphi^{(1)} = ip\varphi^{(1)} = i\psi p = \varphi^{(2)}ip = \varphi^{(2)}f$. Hence $\varphi = (\varphi^{(1)}, \varphi^{(2)})$ is an automorphism of f . It follows from the definitions that $F(\varphi) = \tau$. \square

Lemmas 4.2.3 and 4.2.4 directly imply Theorem 4.2.1. \square

Corollary 4.2.5. *For a filling curve f , the group $\text{Aut}(f)$ is cyclic of finite order dividing $mk_m(f)$ for all $m \geq 1$ where $k_1(f), k_2(f), \dots$ are the numbers defined in Section 2.2.*

5. SELF-TRANSVERSAL CURVES AND SEMICHARTS

5.1. Self-transversal curves. A curve $f : S^1 \rightarrow \Sigma$ is *self-transversal* at $a \in \text{cr}_{>1}(f)$ if for any distinct $x, y \in f^{-1}(a) \subset S^1$ the branches of f at x and y are topologically transversal. The latter condition means that there is a homeomorphism of a neighborhood of a in Σ onto \mathbb{R}^2 whose composition with f sends a neighborhood of x (resp. of y) in S^1 to $\mathbb{R} \times 0$ (resp. to $0 \times \mathbb{R}$). A curve f is *self-transversal* if it is self-transversal at all points of $\text{cr}(f)$.

We now describe the charts of self-transversal curves. Let us say that a chart (n, t) is *straight* if $t(-k) = -t(k)$ for all $k \in \hat{n}$. This property is preserved under conjugation of t by the circular permutation.

Lemma 5.1.1. *A curve is self-transversal if and only if its chart is straight.*

Proof. For a flag $r \in \text{Fl}(f)$ of a curve f , denote the opposite flag by $-r$. Under the identification $\text{Fl}(f) = \bar{n}$ where $n = n(f)$ the involution $r \mapsto -r$ on $\text{Fl}(f)$ corresponds to the negation on \bar{n} . Therefore it suffices to prove that f is self-transversal if and only if $t(-r) = -t(r)$ for all $r \in \text{Fl}(f)$.

Moving around a point $a \in \text{cr}_{>1}(f)$ in the positive direction, we cyclically numerate the flags of f at this point r_1, r_2, \dots, r_{2m} where $m \geq 2$ is the multiplicity of a . For every $q \in \mathbb{Z}/2m\mathbb{Z}$, there is a unique $q' \in \mathbb{Z}/2m\mathbb{Z}$ such that $-r_q = r_{q'}$. The curve f is self-transversal at a if and only if $q' = q + m$ for all q . If f is self-transversal at a , then

$$t(-r_q) = t(r_{q+m}) = r_{q+m+1} = -r_{q+1} = -t(r_q).$$

Conversely, suppose that $t(-r_q) = -t(r_q)$ for all q . Then

$$r_{2q'-q} = t^{q'-q}(r_{q'}) = t^{q'-q}(-r_q) = -t^{q'-q}(r_q) = -r_{q'} = r_q.$$

Therefore $2(q' - q) = 0$. If $q' = q$, then $-r_q = r_{q'} = r_q$ which is impossible. Hence $q' = q + m$. \square

By Theorem 3.2.1 and Lemma 5.1.1, the formula $f \mapsto (n(f), t(f))$ defines a bijective correspondence between self-transversal filling curves considered up to homeomorphism and straight charts considered up to conjugation by the circular permutation.

5.2. Semicharts. Information contained in a straight chart can be packaged in a more compact way. A *semichart* a triple $(n, v : \hat{n} \rightarrow \hat{n}, S \subset \hat{n})$ where $n \geq 1$, v is a bijection, and S meets each orbit of v in an odd number of elements. The group $\mathbb{Z}/n\mathbb{Z}$ acts on the set of semicharts via $(n, v, S) \mapsto (n, \sigma v \sigma^{-1}, \sigma(A))$ where $\sigma = \sigma_n|_{\hat{n}} : \hat{n} \rightarrow \hat{n}$. By convention, there is a unique *empty semichart* (n, v, S) with $n = 0$.

Lemma 5.2.1. *For each $n \geq 0$, there is a $(\mathbb{Z}/n\mathbb{Z})$ -equivariant bijective correspondence between straight charts (n, t) and semicharts (n, v, S) . Under this correspondence $\bar{n}/t = \hat{n}/v$.*

Proof. It suffices to consider the case $n \geq 1$. With a straight chart (n, t) we associate a semichart as follows. For $k \in \hat{n}$, set $v(k) = |t(k)| \in \hat{n}$. Set $S = \{k \in \hat{n} \mid t(k) < 0\}$. We check that (n, v, S) is a semichart. If $v(k) = v(l)$ for $k, l \in \hat{n}$, then either $t(k) = t(l)$ or $t(k) = -t(l) = t(-l)$. Since t is a bijection, $k = l$ or $k = -l$. The latter is impossible since $k, l > 0$. Thus v is injective and therefore bijective. Let $A \subset \hat{n}$ be an orbit

of v . Set $m = \text{card}(A)$ and pick $a \in A$. Then $v^m(a) = a$ and $A = \{a, v(a), \dots, v^{m-1}(a)\}$. An induction on $q = 1, 2, \dots$ shows that $t^q(a) = (-1)^w v^q(a)$ where w is the number of terms of the sequence $a, v(a), \dots, v^{q-1}(a)$ belonging to S . Therefore $t^m(a) = (-1)^w v^m(a) = (-1)^w a$ where $w = \text{card}(A \cap S)$. The equality $t^m(a) = a$ would imply that the orbit of t containing a is contained in the set $\{a, t(a), \dots, t^{m-1}(a)\}$. All its elements are distinct from $-a$ since the absolute values of $t(a), \dots, t^{m-1}(a)$ are equal to $v(a), \dots, v^{m-1}(a)$, all distinct from a . This contradicts the definition of a chart. Hence $t^m(a) = -a$ so that $w = \text{card}(A \cap S)$ is odd. It is clear that the orbits of v are obtained by intersecting the orbits of t with \hat{n} . This gives a bijection $\bar{n}/t = \hat{n}/v$.

Conversely, having a semichart (n, v, S) we define a map $t : \bar{n} \rightarrow \bar{n}$ by $t(k) = v(k)$ for $k \in \hat{n} - S$, $t(k) = -v(k)$ for $k \in S$, and $t(k) = -t(-k)$ for $k \in -\hat{n}$. Then $t(\bar{n}) = \{\pm v(k)\}_{k \in \hat{n}} = \bar{n}$, since $v(\hat{n}) = \hat{n}$. Thus t is bijective. The arguments above show that for any $a \in \hat{n}$, we have $t^m(a) = -a$ where m is the number of elements in the v -orbit of a . Hence (n, t) is a chart. By its very definition, it is straight. It is clear that the arrows from charts to semicharts and backward defined above are mutually inverse. The equivariance of these arrows with respect to the conjugation by the circular permutation is straightforward. \square

Example: the chart $\bar{4} \rightarrow \bar{4}$ defined as the product of two cycles $(1, 3, -1, -3)(2, 4, -2, -4)$ is straight. The associated semichart $(4, v, S)$ is $v = (13)(24)$, $S = \{3, 4\}$.

Using Lemma 5.2.1, we can associate a semichart (considered up to conjugation by the circular permutation) with any self-transversal curve. Combining with the results of Section 5.1, we obtain the following.

Theorem 5.2.2. *There is a bijective correspondence between self-transversal filling curves considered up to homeomorphism and semicharts considered up to conjugation by the circular permutation.*

5.3. Automorphisms. An *automorphism* of a semichart (n, v, S) with $n \geq 1$ is a permutation $\mu : \hat{n} \rightarrow \hat{n}$ such that $\mu v = v \mu$, $\mu(S) = S$, and μ is a power of $\sigma_n|_{\hat{n}} : \hat{n} \rightarrow \hat{n}$. The automorphisms of (n, v, S) form a group with respect to composition denoted $\text{Aut}(v, S)$. By convention, for the empty semichart, this group is trivial. If (n, t) is the straight chart determined by (n, v, S) , then restricting automorphisms of (n, t) to \hat{n} , we obtain that $\text{Aut}(t) = \text{Aut}(v, S)$. This and Theorem 4.2.1 give the following.

Corollary 5.3.1. *For a self-transversal filling curve f with semichart (n, v, S) , we have $\text{Aut}(f) = \text{Aut}(v, S)$.*

Analyzing the automorphisms of a semichart (n, v, S) , it is easy to see that an automorphism preserving an orbit of v setwise has an odd order. We prove the corresponding fact for curves.

Theorem 5.3.2. *Let f be a self-transversal filling curve and $a \in \text{cr}(f)$. Let $\text{Aut}_a(f)$ be the subgroup of $\text{Aut}(f)$ consisting of the (isotopy classes of) automorphisms of f preserving a . Then $\text{Aut}_a(f)$ is a finite cyclic group of odd order dividing the multiplicity $m = m_a$ of a .*

Proof. Moving around a in the positive direction, we cyclically numerate r_1, r_2, \dots, r_{2m} the flags of f at a . Let j be the minimal element of the set $\{1, 2, \dots, 2m\}$ such that there is $\varphi \in \text{Aut}_a(f)$ with $\varphi_*(r_1) = r_{1+j}$. Then $\varphi_*(r_q) = r_{q+j}$ for all $q \in \mathbb{Z}/2m\mathbb{Z}$ and any $\psi \in \text{Aut}_a(f)$ is a power of φ . Indeed, $\psi_*(r_1) = r_q$ for some q . If $q \not\equiv 1 \pmod{j}$ then a product of ψ_* with a power of φ_* transforms r_1 into r_{1+k} with $1 \leq k < j$ which contradicts the choice of j . If $q = 1 + kj$ with $k \in \mathbb{Z}$, then $\varphi_*^{-k} \psi_*(r_1) = r_1$ and therefore $\psi = \varphi^k$.

By the choice of j , the residues $1 + 2j, 1 + 3j, \dots \pmod{2m}$ do not take values $2, 3, \dots, j - 1$. This is possible only if $dj = 0 \pmod{2m}$ for some $d \geq 2$. Take the smallest such d . Then $\varphi_*^d(r_1) = r_{1+dj} = r_1$. Hence $\varphi^d = 1$. It remains to prove that d is odd. If $d = 2e$ with $e \in \mathbb{Z}$, then $ej = m \pmod{2m}$ and $\varphi_*^e(r_1) = r_{1+ej} = r_{1+m}$. Since f is self-transversal, the flag r_{1+m} is opposite to r_1 . This contradicts the equality $\varphi_*^e(r_1) = r_{1+m}$ since automorphisms of curves cannot transform incoming flags into outgoing ones. \square

Remark 5.3.3. It is easy to see that for any curve f and any $a \in \text{cr}(f)$ of multiplicity $m \geq 1$, we have $|\text{Aut}(f)/\text{Aut}_a(f)| \leq k_m(f)$. By Theorem 5.3.2, if f is self-transversal and filling, then $|\text{Aut}_a(f)|$ is an odd divisor of m . In particular, if m is a power of 2, then $\text{Aut}_a(f) = 1$. Under further assumptions on f , these observations may give $\text{Aut}(f) = 1$. For example, if $k_{2q}(f) = 1$ for some $q \in \mathbb{Z}$, then $\text{Aut}(f) = 1$. Another example: if $k_5(f) = 4$ and $k_6(f) = 1$, then Corollary 4.2.5 and Theorem 5.3.2 imply that $\text{Aut}(f) = 1$.

5.4. Coxeter groups. The permutations $t : \bar{n} \rightarrow \bar{n}$ such that $t(-k) = -t(k)$ for all $k \in \bar{n}$ form a group W_n . This is the Coxeter group of type B , see for instance [Bo], [BB]. By the results above, the charts of self-transversal curves are elements of W_n for an appropriate n . (To avoid the circular indeterminacy in the definition of the charts, we assume the curves to be pointed, cf. Section 7.1.) Conversely, every element of W_n whose orbits in \bar{n} are negation invariant gives rise to a pointed self-transversal filling curve on a surface of a certain genus. Other elements of W_n give rise to pointed self-transversal filling curves on singular surfaces

with simple singularities, cf. Section 3.7. This somewhat surprising connexion between curves and Coxeter groups of type B can be exploited to study curves. For example, for any pair of pointed self-transversal curves (possibly lying on different surfaces but yielding the same number n), we can consider the Kazhdan-Lusztig polynomial of their charts. It seems however that it is quite difficult to compute this polynomial directly from the curves. This is due to a poor connexion between multiplication in W_n and the topology of curves.

6. COUNTING CURVES

6.1. Preliminaries. We begin with a few simple remarks on group actions. The set of orbits of a (left) action of a group G on a set \mathcal{S} is denoted \mathcal{S}/G . For $a \in \mathcal{S}$, let $\text{Stab}(a) = \{g \in G \mid ga = a\}$ be the stabilizer of a . For an orbit of this action $A \subset \mathcal{S}$, set $\text{Stab}_A = \text{Stab}(a)$ for some $a \in A$. The isomorphism class of the group Stab_A does not depend on the choice of a : if $a' \in A$, then there is $g \in G$ such that $a' = ga$ and $\text{Stab}(a') = g \text{Stab}(a) g^{-1}$. The number of elements of a finite group G will be denoted $|G|$.

Lemma 6.1.1. *Let G be a finite group acting on a finite set \mathcal{S} . Then*

$$(6.1.1) \quad \text{card}(\mathcal{S}) = |G| \sum_{A \in \mathcal{S}/G} \frac{1}{|\text{Stab}_A|}.$$

Proof. For the orbit $A \subset \mathcal{S}$ of $a \in \mathcal{S}$, the formula $g \mapsto ga$ defines a surjection $G \rightarrow A$ and a bijection $G/\text{Stab}(a) \approx A$. Hence $\text{card}(A) = |G|/|\text{Stab}(a)| = |G|/|\text{Stab}_A|$. Summing up over all the orbits, we obtain

$$\text{card}(\mathcal{S}) = \sum_{A \in \mathcal{S}/G} \text{card}(A) = \sum_{A \in \mathcal{S}/G} |G|/|\text{Stab}_A| = |G| \sum_{A \in \mathcal{S}/G} 1/|\text{Stab}_A|.$$

□

6.2. Counting filling curves. A sequence of integers $K = (k_1, k_2, \dots)$ is *finite* if $k_m = 0$ for all sufficiently big m . For a finite sequence K , set $n(K) = \sum_{m \geq 1} m k_m$. By convention, $(-1)! = 0! = 1$.

Theorem 6.2.1. *Let $K = (k_1, k_2, \dots)$ be a finite sequence of non-negative integers. Let $\mathcal{C}(K)$ be the set of homeomorphism classes of filling curves f such that $k_m(f) = k_m$ for all $m \geq 1$. Then*

$$(6.2.1) \quad \sum_{f \in \mathcal{C}(K)} \frac{1}{|\text{Aut}(f)|} = (n(K) - 1)! \prod_{m \geq 1} \frac{1}{k_m!} \left(\frac{(2m-1)!}{m!} \right)^{k_m}.$$

Proof. If $k_1 = k_2 = \dots = 0$, then both sides of this formula are equal to 1. Assume that at least one k_m is non-zero so that $n = n(K) \geq 1$. Let \mathcal{S} be the set of charts (n, t) such that t has k_m orbits of cardinality $2m$ for all $m \geq 1$. The set \mathcal{S} is invariant under the action of $G = \mathbb{Z}/n\mathbb{Z}$ on charts defined in Section 3.2. By Theorem 3.2.1, assigning to $f \in \mathcal{C}(K)$ its chart, we obtain $\mathcal{C}(K) = \mathcal{S}/G$. By Theorem 4.2.1, the stabilizer of the chart of f is isomorphic to $\text{Aut}(f)$. Hence Formula 6.1.1 gives

$$(6.2.2) \quad \text{card}(\mathcal{S}) = n \sum_{f \in \mathcal{C}(K)} \frac{1}{|\text{Aut}(f)|}.$$

We now compute $\text{card}(\mathcal{S})$. To specify $t \in \mathcal{S}$ we need to specify a partition of \bar{n} into the orbits of t and the action of t on these orbits. Since the orbits of t are invariant under the negation, the partitions of \bar{n} in question bijectively correspond to splittings of the set $\hat{n} = \{1, 2, \dots, n\}$ into $k_1 + k_2 + \dots$ disjoint subsets such that k_1 of them have 1 element, k_2 of them have 2 elements, etc. The number of such splittings of \hat{n} is

$$(6.2.3) \quad \frac{n!}{\prod_{m \geq 1} (m!)^{k_m} k_m!}.$$

The number of transitive actions of t on a set of $M \geq 2$ elements is $(M-1)!$. Therefore

$$\text{card}(\mathcal{S}) = \frac{n!}{\prod_{m \geq 1} (m!)^{k_m} k_m!} \prod_{m \geq 1} ((2m-1)!)^{k_m} = n! \prod_{m \geq 1} \frac{1}{k_m!} ((2m-1)!/m!)^{k_m}.$$

This equality and Formula 6.2.2 imply Formula 6.2.1. □

Corollary 6.2.2. *If under the conditions of Theorem 6.2.1, $\gcd\{m k_m\}_{m \geq 1} = 1$, then*

$$\text{card} \mathcal{C}(K) = (n(K) - 1)! \prod_{m \geq 1} \frac{1}{k_m!} ((2m-1)!/m!)^{k_m}.$$

This follows directly from Corollary 4.2.5 and Theorem 6.2.1. As an illustration consider a few special cases. In the case where $k_m = 0$ for all $m \geq 3$, we have $n = k_1 + 2k_2$ and Theorem 6.2.1 gives

$$(6.2.4) \quad \sum_{f \in \mathcal{C}(K)} \frac{1}{|\text{Aut}(f)|} = \frac{(k_1 + 2k_2 - 1)!}{k_1! k_2!} 3^{k_2}.$$

For $k_1 = 0$, the right-hand side here simplifies to $((2k_2 - 1)!/k_2!) 3^{k_2}$. For $k_2 = 0$, the right-hand side of Formula 6.2.4 simplifies to $1/k_1$. The resulting formula can be verified directly since the set $\mathcal{C}(k_1, 0, 0, \dots)$ consists of one element f that is an embedding $S^1 \hookrightarrow S^2$ with k_1 corners and $\text{Aut}(f) = \mathbb{Z}/k_1\mathbb{Z}$.

Formula 6.2.1 can be rewritten as an equality in the ring $\mathbb{Q}[[t_1, t_2, \dots]]$ of formal power series in commuting variables t_1, t_2, \dots with rational coefficients. Let \mathcal{C} be the set of homeomorphism classes of filling curves. Then

$$(6.2.5) \quad \sum_{f \in \mathcal{C}} \frac{1}{(n(f) - 1)! |\text{Aut}(f)|} t_1^{k_1(f)} t_2^{k_2(f)} t_3^{k_3(f)} \dots = \prod_{m \geq 1} \exp \left(\frac{(2m - 1)!}{m!} t_m \right).$$

Here the product on the right hand side is a limit of finite products $\prod_{m \geq 1}^N$ when $N \rightarrow \infty$. A typical monomial $t_1^{k_1} t_2^{k_2} \dots t_q^{k_q}$ appears in this limit with the same coefficient as in $\prod_{m \geq 1}^q$.

Theorem 6.2.3. *Let $K = (k_1, k_2, \dots)$ be a finite sequence of non-negative integers. Let $\mathcal{C}_{str}(K)$ be the set of homeomorphism classes of self-transversal filling curves f such that $k_m(f) = k_m$ for all $m \geq 1$. Then*

$$(6.2.6) \quad \sum_{f \in \mathcal{C}_{str}(K)} \frac{1}{|\text{Aut}(f)|} = (n(K) - 1)! \prod_{m \geq 1} \frac{1}{k_m!} \left(\frac{2^{m-1}}{m} \right)^{k_m}.$$

Proof. The proof goes along the same lines as the proof of Theorem 6.2.1 except that here we count semicharts. To specify a semichart (n, v, S) we specify a partition of \hat{n} into the orbits of v , the action of v on these orbits, and the intersections of S with the orbits. The number of partitions of \hat{n} into the orbits of v is given by Formula 6.2.3. An induction on m shows that a set of m elements contains 2^{m-1} subsets having an odd number of elements. Therefore the number of semicharts (n, v, S) is equal to

$$\frac{n!}{\prod_{m \geq 1} (m!)^{k_m} k_m!} \prod_{m \geq 1} (2^{m-1} (m-1)!)^{k_m} = n! \prod_{m \geq 1} \frac{1}{k_m!} (2^{m-1}/m)^{k_m}.$$

The rest of the argument is as in the proof of Theorem 6.2.1. □

Corollary 6.2.4. *If $\gcd(\{mk_m\}_{m \geq 1}) = 1$ of $k_{2^q} = 1$ for some $q = 0, 1, 2, \dots$, then*

$$\text{card } \mathcal{C}_{str}(K) = (n(K) - 1)! \prod_{m \geq 1} \frac{1}{k_m!} (2^{m-1}/m)^{k_m}.$$

Theorem 6.2.3 implies Formulas 1.0.1 and 1.0.2 of the introduction.

7. FURTHER CLASSES OF CURVES

7.1. Pointed curves. A *pointed curve* is a curve f endowed with a distinguished point $x \in S^1 - \text{sing}(f)$ called the *base point*. It should be stressed that x is not viewed as a corner of f . Using x as the starting point in the constructions of Section 3.4, we obtain a canonical identification $\text{Fl}(f) = \bar{n}$ where $n = n(f) = \text{card}(\text{sing}(f))$. Therefore with a pointed curve we can associate a chart without any indeterminacy. A *homeomorphism* of pointed curves is a homeomorphism of curves mapping the base point to the base point. It is clear that homeomorphic pointed curves have the same charts. This yields a bijective correspondence between pointed filling curves considered up to homeomorphism and charts. Similarly, there is a bijective correspondence between self-transversal pointed filling curves considered up to homeomorphism and semicharts.

Note that the group of isotopy classes of automorphisms of a pointed filling curve is trivial.

7.2. Generic curves. A curve is *generic* if it is self-transversal and all its crossings have multiplicity 2. We allow generic curves to have corners. It is obvious that a curve is generic if and only if its chart (n, t) is straight and either $n = 0$ or $n \geq 1$ and every orbit of t consists of 2 or 4 elements. A self-transversal curve is generic if and only if either $n = 0$ or $n \geq 1$ and the map $v : \hat{n} \rightarrow \hat{n}$ in its semichart (n, v, S) is an involution. We call such semicharts *involutive*. For an involutive semichart (n, v, S) , the condition that S meets each orbit of v in an odd number of elements means simply that S meets every orbit of v in one element. Theorem 5.2.2 yields a bijective correspondence between generic filling curves considered up to homeomorphism and involutive semicharts considered up to conjugation by the circular permutation.

7.3. Alternating curves. A curve f is *alternating* if the flag rotation $t : \text{Fl}(f) \rightarrow \text{Fl}(f)$ transforms incoming flags into outgoing ones. Since the number of incoming and outgoing flags of f is the same, t then transforms outgoing flags into incoming ones. A generic curve is alternating if and only if it has no crossings. It is obvious that a curve with chart (n, t) is alternating if and only if either $n = 0$ or $n \geq 1$ and $t(\hat{n}) = -\hat{n}$.

A self-transversal curve f with semichart (n, v, S) is alternating if and only if $n = 0$ or $S = \hat{n}$. The condition that $S = \hat{n}$ meets each orbit of v in an odd number of elements means simply that each orbit of v has an odd number of elements. In other words, v must be a permutation of odd order. Theorem 5.2.2 yields a bijective correspondence between alternating self-transversal filling curves considered up to homeomorphism and permutations $\{\hat{n} \rightarrow \hat{n}\}_{n \geq 0}$ of odd order considered up to conjugation by the circular permutation. (By convention, for $n = 0$, there is one permutation $\hat{n} \rightarrow \hat{n}$ of odd order.)

7.4. Beaming curves. The notion of a beaming curve is in a sense opposite to the one of an alternating curve. A curve $f : S^1 \rightarrow \Sigma$ is *beaming at a crossing* $a \in \text{cr}_{>1}(f)$ if one can draw a line on Σ through a such that the incoming flags of f at a lie on one side of this line and the outgoing flags of f at a lie on its other side. We can rephrase this condition in terms of the flag rotation $t : \text{Fl}(f) \rightarrow \text{Fl}(f)$ by saying that there is only one incoming flag r at a such that $t(r)$ is outgoing (equivalently, there is only one incoming flag r at a such that $t^{-1}(r)$ is outgoing). The curve f is *beaming* if it is beaming at all its crossings. An alternating curve is beaming if and only if it has no crossings. All generic curves are beaming.

It is clear that a curve with chart (n, t) is beaming if and only if either $n = 0$ or $n \geq 1$ and each orbit of $t : \bar{n} \rightarrow \bar{n}$ contains only one element $r > 0$ such that $t(r) < 0$. A self-transversal curve with semichart (n, v, S) is beaming if and only if either $n = 0$ or S meets every orbit of v in one element.

7.5. Coherent curves. Consider a self-transversal curve f and a crossing $a \in \text{cr}_{>1}(f)$. A *branch of f at a* is a union of two opposite flags at a . The curve f has m_a pairwise transversal branches at a . Moving on the ambient surface in the positive direction around a we obtain a cyclic order on the set, B_a , of branches of f at a . On the other hand, starting at a generic point on the curve and traversing the whole curve we go once along each branch of f at a . This also yields a cyclic order on B_a . The curve f is *coherent at a* if these two cyclic orders coincide. A curve is *coherent* if it is self-transversal and coherent at all crossings. Since a set of two elements has only one cyclic order, all generic curves are coherent. The curve in Example 2.4.3 is coherent and non-generic. Note that if a coherent curve is non-generic, then inverting orientation on the curve (or on the ambient surface) we obtain a non-coherent curve.

A semichart (n, v, S) with $n \geq 1$ is *coherent* if for any $k \in \hat{n}$ either $k < v(k)$ or k is the maximal element in its v -orbit and then $v(k)$ is the minimal element in the v -orbit of k . By convention, the empty semichart is coherent. The coherency of a semichart is preserved under conjugation by the circular permutation. All involutive semicharts are coherent.

Theorem 7.5.1. *A self-transversal curve is coherent if and only if its semichart is coherent. This gives a bijective correspondence between coherent filling curves considered up to homeomorphism and coherent semicharts considered up to conjugation by the circular permutation.*

Proof. Let f be a self-transversal curve with chart (n, t) and semichart (n, v, S) . Pick $a \in \text{cr}_{>1}(f)$ and identify $\text{Fl}(f) = \bar{n}$ as above. Let $k_1 < k_2 < \dots < k_m$ be elements of \hat{n} corresponding to the outgoing flags at a . The curve f is coherent at a if and only if $t(k_i) = \pm k_{i+1}$ for $i = 1, \dots, m-1$ and $t(k_m) = \pm k_1$. This holds iff $v(k_i) = k_{i+1}$ for $i = 1, \dots, m-1$ and $v(k_m) = k_1$. Therefore f is coherent at all its crossings iff (n, v, S) is coherent. The second claim of the theorem follows from the first claim and Theorem 5.2.2. \square

7.6. Perfect curves. A pointed curve f is *perfect* if it is self-transversal, beaming, coherent, and satisfies the following property: (*) starting at the base point and moving along the curve we enter each crossing $a \in \text{cr}_{>1}(f)$ for the first time along the unique incoming flag s_a at a such that the flag $t^{-1}(s_a)$ is outgoing.

Under these assumptions, the outgoing flags of f at a can be numerated $r_1 = -s_a, r_2, \dots, r_m$ so that moving around a we encounter consecutively $r_1, r_2, \dots, r_m, -r_1, -r_2, \dots, -r_m$. (Here $t^{-1}(s_a) = r_m$.) Note that if a curve is self-transversal, beaming, and coherent, then it is always possible to choose a base point on it to satisfy $(*)$ at any given crossing. A perfect curve satisfies $(*)$ for all crossings and one and the same base point. Trivial curves and trivial curves with corners are perfect.

A semichart (n, v, S) is *perfect* if it is coherent and either $n = 0$ or S meets every orbit of $v : \hat{n} \rightarrow \hat{n}$ in one element which is the maximal element of this orbit with respect to the standard order on $\hat{n} \subset \mathbb{R}$.

Theorem 7.6.1. *A pointed curve is perfect iff its semichart is perfect. This gives a bijective correspondence between perfect pointed filling curves considered up to homeomorphism and perfect semicharts.*

Proof. The first claim follows from the results above and the following observation. Consider the chart (n, t) of a pointed curve f and the t -orbit $A \subset \overline{n}$ corresponding to $a \in \text{cr}_{>1}(f)$. Condition $(*)$ can be interpreted by saying that the only element $r \in A \cap \hat{n}$ such that $t(r) < 0$ is the maximal element of $A \cap \hat{n}$. The second claim of the theorem follows directly from the first claim. \square

7.7. Remark. The technique of charts yield formulas counting alternating/beaming/coherent/perfect curves with weights as in Section 6. We state these formulas for pointed curves. Let \mathcal{C}_p be the set of homeomorphism classes of pointed filling curves and $\mathcal{C}_{p, \text{str}}$ be the set of homeomorphism classes of self-transversal pointed filling curves. Then

$$\sum_{f \in \mathcal{C}_p} \frac{1}{n(f)!} t_1^{k_1(f)} t_2^{k_2(f)} \dots = \prod_{m \geq 1} \exp \left(\frac{(2m-1)!}{m!} t_m \right),$$

$$\sum_{f \in \mathcal{C}_{p, \text{str}}} \frac{1}{n(f)!} t_1^{k_1(f)} t_2^{k_2(f)} \dots = \prod_{m \geq 1} \exp \left(\frac{2^{m-1}}{m} t_m \right).$$

8. WORDS

8.1. Words and their automorphisms. An *alphabet* is a finite set and *letters* are its elements. A (*signed*) *word of length* $n \geq 1$ in an alphabet E is a pair $(w : \hat{n} \rightarrow E, S)$ where $\hat{n} = \{1, 2, \dots, n\}$. Such a word is encoded by the sequence $w_1^S w_2^S \dots w_n^S$ where $w_k^S = w(k)$ for $k \in S$ and $w_k^S = (w(k))^+$ for $k \in \hat{n} - S$. For example, the sequence AB^+A^+ in the alphabet $E = \{A, B\}$ encodes the map $\hat{3} \rightarrow E$ sending $1, 2, 3$ to A, B, A and the set $S = \{1\}$. More standard unsigned words appear when $S = \hat{n}$.

Recall the map $\sigma = \sigma_n : \hat{n} \rightarrow \hat{n}$ sending k to $k+1$ for $k = 1, 2, \dots, n-1$ and sending n to 1 . The *circular permutation* of a word (w, S) is the word $(w\sigma^{-1}, \sigma(S))$. This transforms $w_1^S w_2^S \dots w_n^S$ into $w_n^S w_1^S w_2^S \dots w_{n-1}^S$. A word (w, S) is *full* if w is surjective, i.e., if all letters appears in $w_1^S w_2^S \dots w_n^S$ at least once (possibly with $+$).

The basic equivalence relation in the class of words is *congruence*. A word $(w : \hat{n} \rightarrow E, S)$ is *congruent* to a word $(w' : \hat{n}' \rightarrow E', S')$ if $n = n', S = S'$, and there is a bijection $\psi : E \rightarrow E'$ such that $\psi w = w'$. For example, the word ABA^+ in the alphabet $\{A, B\}$ is congruent to the word CDC^+ in the alphabet $\{C, D\}$. It is easy to classify words up to congruence. Observe that a mapping $w : \hat{n} \rightarrow E$ gives rise to an equivalence relation on \hat{n} called *w-equivalence*: $k, l \in \hat{n}$ are *w-equivalent* if $w(k) = w(l)$. If $w(\hat{n}) = E$ then the set of *w-equivalence* classes can be identified with E by assigning to a *w-equivalence* class its image in E . It follows from the definitions that two full words $(w, S), (w', S')$ of the same length n are congruent if and only if $S = S'$ and the relations of *w-equivalence* and *w'-equivalence* on \hat{n} coincide.

An *automorphism* of a word $W = (w : \hat{n} \rightarrow E, S)$ is a pair $(\psi : E \rightarrow E, m \in \mathbb{Z}/n\mathbb{Z})$ such that $\psi w = w\sigma^m : \hat{n} \rightarrow E$ and $\sigma^m(S) = S$. These conditions can be rephrased by saying that the sequence $(\psi w)_1^S (\psi w)_2^S \dots (\psi w)_n^S$ can be obtained from $w_1^S w_2^S \dots w_n^S$ by the $(-m)$ -th power of the circular permutation. The automorphisms of W form a group $\text{Aut}(W)$ with unit $(\text{id}, 0)$ and multiplication $(\psi, m)(\psi', m') = (\psi\psi', m+m')$. This group is preserved under circular permutations of W . The formula $(\psi, m) \mapsto m$ defines a “forgetting” group homomorphism $p : \text{Aut}(W) \rightarrow \mathbb{Z}/n\mathbb{Z}$.

We formally introduce a unique *empty word* of length 0 in an empty alphabet. By convention, this word is full and its group of automorphisms is trivial.

Lemma 8.1.1. *Let $W = (w, S)$ be a full word of length $n \geq 1$ in an alphabet E . Then*

- (i) *the homomorphism $p : \text{Aut}(W) \rightarrow \mathbb{Z}/n\mathbb{Z}$ is injective so that $\text{Aut}(W)$ is a finite cyclic group;*
- (ii) *the order of $\text{Aut}(W)$ divides $\gcd(\{m k_m\}_{m \geq 1})$ where k_m is the number of letters of E appearing in w (with or without $+$) exactly m times.*

Proof. If $(\psi, m) \in \text{Ker } p$, then $m = 0$ and $\psi w = w$. Since $w(\hat{n}) = E$, we have $\psi = \text{id}$. Hence p is injective.

Claim (ii) is proven in the same way as Lemma 4.1.1 with $M = \{k \in \hat{n} \mid \text{card } w^{-1}(w(k)) = m\}$. \square

Example 8.1.2. For the word $W = ABAB$ in the alphabet $E = \{A, B\}$, the group $\text{Aut}(W) = \mathbb{Z}/4\mathbb{Z}$ is generated by $(\psi : E \rightarrow E, m = 1)$ where ψ permutes A and B . For $W = A^+B^+AB$, we have $\text{Aut}(W) = 1$. For the word $W = ABACAD$ in the alphabet $E = \{A, B, C, D\}$, the group $\text{Aut}(W) = \mathbb{Z}/3\mathbb{Z}$ is generated by $(\psi : E \rightarrow E, m = 2)$ where ψ fixes A and sends B, C, D to C, D, B , respectively.

8.2. Words and charts. A chart (n, t) gives rise to a word $W(t) = (w, S)$ as follows. If $n = 0$, then $W(t) = \emptyset$. For $n \geq 1$, the mapping w is the composition of the inclusion $\hat{n} \hookrightarrow \bar{n}$ with the projection $\bar{n} \rightarrow \bar{n}/t$. The set $S \subset \hat{n}$ consists of all $k \in \hat{n}$ such that $t(k) < 0$. Then $W(t) = (w, S)$ is a full word in the alphabet \bar{n}/t . It is clear that $W(\sigma_n t(\sigma_n)^{-1})$ is obtained from $W(t)$ by the circular permutation.

Lemma 8.2.1. *For any chart (n, t) , there is a canonical group injection $\text{Aut}(t) \hookrightarrow \text{Aut}(W(t))$.*

Proof. For $n = 0$, both groups are trivial. Let $n \geq 1$ and $W(t) = (w, S)$. Pick $\varphi \in \text{Aut}(t)$. Recall that $\varphi = \sigma^m$ where $\sigma = \sigma_n : \bar{n} \rightarrow \bar{n}$ and $m \in \mathbb{Z}/n\mathbb{Z}$. For $k \in \bar{n}$, denote its t -orbit by $[k]$. The equality $\varphi t = t\varphi$ implies that if $k, l \in \bar{n}$ lie in the same orbit of t , that is if $[k] = [l]$, then $[\varphi(k)] = [\varphi(l)]$. Thus φ induces a map $\psi : \bar{n}/t \rightarrow \bar{n}/t$. Since φ is onto, so is ψ . Hence ψ is a bijection. We verify that $\psi w = w\sigma^m$: for $k \in \hat{n}$,

$$\psi(w(k)) = \psi([k]) = [\varphi(k)] = [\sigma^m(k)] = w(\sigma^m(k)).$$

The equality $t\sigma^m(k) = \sigma^m t(k)$ implies that $t\sigma^m(k) < 0$ iff $t(k) < 0$. Thus $\sigma^m(S) = S$ and $(\psi, m) \in \text{Aut}(W)$. The homomorphism $\text{Aut}(t) \rightarrow \text{Aut}(W)$, $\varphi \mapsto (\psi, m)$ is injective: if $m = 0$, then $\varphi = \sigma^0 = \text{id}$. \square

Example 8.2.2. The charts $t_1, t_2, t_3 : \bar{4} \rightarrow \bar{4}$ defined in Section 4.1 have the same orbits $A = \{-3, -1, 1, 3\}$ and $B = \{-4, -2, 2, 4\}$. We have $W(t_1) = A^+B^+AB$ and $W(t_2) = W(t_3) = ABAB$. Thus, different charts may yield the same word and in general $\text{Aut}(t) \neq \text{Aut } W(t)$.

8.3. Words associated with curves. Any pointed curve f gives rise to a word $W(f)$ in the alphabet $\text{cr}(f)$. If f is a trivial curve, then $W(f) = \emptyset$. For a non-trivial f , the word $W(f)$ is obtained by first taking the chart (n, t) of f and then taking the associated word in the alphabet $\bar{n}/t = \text{cr}(f)$. By the results above,

$$\text{Aut}(f) = \text{Aut}(t) \subset \text{Aut}(W(f)) \subset \mathbb{Z}/n\mathbb{Z}.$$

It is easy to read the word $W(f)$ directly from f . Label all points of $\text{cr}(f)$ with distinct letters (the resulting set of letters is identified with $\bar{n}/t = \text{cr}(f)$). Label each point $x \in \text{sing}(f)$ with the letter labelling $f(x)$. Traverse S^1 counterclockwise starting from the base point in $S^1 - \text{sing}(f)$ and write down consecutively the letters appearing when we cross $\text{sing}(f)$. Moreover, crossing a point $x \in \text{sing}(f)$ provide the corresponding letter with the superscript $+$ if the flag obtained from $(x, +)$ by the flag rotation is outgoing. This gives $W(f)$. The word $W(f)$ generalizes the Gauss word of a generic curve.

Similar constructions apply to non-pointed curves. Their charts are defined up to conjugation by circular permutations and their words are defined up to circular permutations.

Example 8.2.2 yields different curves with the same associated word. We now address the realization problem for words. We say that a word W is *realized* by a curve f if W is congruent to $W(f)$.

Theorem 8.3.1. *Every full word can be realised by a pointed filling curve.*

Proof. Let $W = (w, S)$ be a full word of length n in an alphabet E . It suffices to realize W as the word of a chart. Pick a w -equivalence class $A \subset \hat{n}$. Let a_1, \dots, a_q be the elements of $A - (A \cap S)$ numerated in an arbitrary way. Let a_{q+1}, \dots, a_{q+r} be the elements of $A \cap S$. We cyclically order the set $\pm A = \{\pm a\}_{a \in A}$ by

$$a_1 \prec a_2 \prec \dots \prec a_{q+1} \prec -a_{q+1} \prec a_{q+2} \prec -a_{q+2} \prec \dots \prec a_{q+r} \prec -a_{q+r} \prec -a_q \prec -a_{q-1} \prec \dots \prec -a_1 \prec a_1.$$

Let $t : \pm A \rightarrow \pm A$ be the map sending each element to its immediate follower. Applying this procedure to all w -equivalence classes $A \subset \hat{n}$ we obtain a chart $(n, t : \bar{n} \rightarrow \bar{n})$ with orbits $\{\pm A\}_A$. The set \bar{n}/t can be identified with the set of w -equivalence classes in \hat{n} , that is with E . Under these identifications $W(t) = W$. \square

8.4. Words of self-transversal curves. Similarly to charts, each semichart (n, v, S) gives rise to a word $W(v, S) = (w : \hat{n} \rightarrow \hat{n}/v, S)$ where w is the natural projection from \hat{n} to the set of v -orbits. We have $W(v, S) = W(t)$ where (n, t) is the straight chart determined by (n, v, S) . Therefore for a self-transversal pointed curve f with chart (n, t) and semichart (n, v, S) , we have $W(f) = W(t) = W(v, S)$. The definition of a semichart implies that the word $W(v, S) = (w, S)$ is *odd* in the following sense: the intersection of S with any w -equivalence class in \hat{n} has an odd number of elements.

8.5. Words of coherent curves. We show that odd words bijectively correspond to coherent curves.

Theorem 8.5.1. *Any odd full word can be realised by a pointed coherent filling curve. This curve is unique up to homeomorphism.*

Proof. By Theorem 7.5.1, it suffices to show that any odd full word $W = (w, S \subset \hat{n})$ arises from a unique coherent semichart. We define a permutation $v = v_w : \hat{n} \rightarrow \hat{n}$ as follows. For $k \in \hat{n}$, let $v(k)$ be the minimal element of the set $\{k+1, k+2, \dots, n\}$ that is w -equivalent to k . If there are no elements in the latter set w -equivalent to k , let $v(k)$ be the minimal element of the set $\{1, 2, \dots, k\}$ that is w -equivalent to k . In particular, $v(k) = k$ iff $w^{-1}(w(k)) = \{k\}$. The resulting mapping $v : \hat{n} \rightarrow \hat{n}$ is bijective. Its orbits are the w -equivalence classes in \hat{n} . The triple (n, v, S) is a coherent semichart whose associated word is W . It follows from the definitions that this is the only such semichart. \square

Corollary 8.5.2. *The formula $f \mapsto W(f)$ defines a bijective correspondence between pointed (resp. non-pointed) coherent filling curves considered up to homeomorphism and odd full words considered up to congruence (resp. considered up to congruence and conjugation by the circular permutation).*

For an odd full word W in an alphabet E , the crossings and corners of the corresponding coherent curve are labeled (in a 1-to-1 way) by elements of E . This set of points has a natural order: the point labeled with $a \in E$ precedes the point labeled with $b \in E$ if the letter a appears in W before b . This order of course is not preserved under circular permutations of W .

The next theorem shows that a coherent filling curve has as many symmetries as its associated word.

Theorem 8.5.3. *For any (non-pointed) coherent filling curve f , we have $\text{Aut}(f) = \text{Aut}(W(f))$.*

Proof. Let (n, v, S) be the semichart of f and $W = (w, S)$ be the associated word. Since (n, v, S) is coherent, the map $v = v_w : \hat{n} \rightarrow \hat{n}$ is computed from w as in the proof of the previous theorem. As we know, $\text{Aut}(f) = \text{Aut}(v, S) \subset \text{Aut}(W)$. We have to verify that every automorphism $(\psi, m \in \mathbb{Z}/n\mathbb{Z})$ of W lies in $\text{Aut}(v, S)$. It suffices to show that m belongs to the image of $\text{Aut}(v, S)$ under the inclusions $\text{Aut}(v, S) \subset \text{Aut}(W) \subset \mathbb{Z}/n\mathbb{Z}$. Thus, we need to check that the m -th power of the circular permutation $\sigma : \hat{n} \rightarrow \hat{n}$ commutes with v and keeps S setwise. The last condition follows from the definition of an automorphism of W . The equality $\psi w = w \sigma^m$ implies that if $k, l \in \hat{n}$ are w -equivalent then $\sigma^m(k), \sigma^m(l)$ are w -equivalent. Thus the map $\sigma^m : \hat{n} \rightarrow \hat{n}$ sends w -equivalence classes to w -equivalence classes. Since this map also preserves the cyclic order in \hat{n} , it must commute with v . \square

Theorem 8.5.1 yields for any odd word $W = (w, S \subset \hat{n})$ a coherent filling curve on a surface. The genus of this surface denoted $g(W)$ is a fundamental geometric invariant of W . We give an explicit formula for $g(W)$.

We shall use the symbol \prec for the standard cyclic order on \hat{n} . Thus for $k, l, m \in \hat{n}$, we have $k \prec l \prec m$ if $l \neq k, l \neq m$ and increasing k by $1, 2, \dots$, we obtain first $l \pmod{n}$ and then $m \pmod{n}$. For $k, l \in \hat{n}$, set $|k, l| = \{r \in \hat{n} \mid k \prec r \prec l\}$. In particular, $|k, k| = \hat{n} - \{k\}$.

For $k \in \hat{n}$, set $d_S(k) = 1$ if $k \in \hat{n} - S$ and $d_S(k) = 0$ if $k \in S$. Set $k^+ = v_w(k)$, where $v_w : \hat{n} \rightarrow \hat{n}$ is the bijection in the proof of Theorem 8.5.1. For $k, l \in \hat{n}$, set $\langle k, l \rangle = 1$ if $k \neq l$ and k is w -equivalent to l ; in all the other cases $\langle k, l \rangle = 0$. Set

$$\Delta_{k,l}^0 = \begin{cases} d_S(k) + d_S(l) + 1, & \text{if } k \prec l \prec k^+ \prec l^+ \prec k \text{ or } l \prec k \prec l^+ \prec k^+ \prec l, \\ \langle k, l \rangle d_S(k) d_S(l), & \text{otherwise.} \end{cases}$$

Finally, define a residue $W_{k,l} \in \mathbb{Z}/2\mathbb{Z}$ by

$$(8.5.1) \quad W_{k,l} = \sum_{q \in |k, k^+|, r \in |l, l^+|} \langle q, r \rangle + d_S(k) \sum_{r \in |l, l^+|, r \neq k^+} \langle k, r \rangle + d_S(l) \sum_{q \in |k, k^+|, q \neq l^+} \langle q, l \rangle + \Delta_{k,l}^0 \pmod{2}.$$

Theorem 8.5.4. *For any odd full word W of length n , we have $g(W) = (1/2) \text{rank}(W_{k,l})_{k,l \in \hat{n}}$ where rank is the usual rank of a square $(n \times n)$ -matrix over the field $\mathbb{Z}/2\mathbb{Z}$.*

Theorem 8.5.4 will be proven in Section 9. Note that $W_{k,k} = 0$ for all $k \in \hat{n}$ and $W_{k,l} = d_S(k) d_S(l)$ for distinct w -equivalent $k, l \in \hat{n}$. If the alphabet consists of only one letter, all elements of \hat{n} are w -equivalent and $g(W) = (n - \text{card}(S))/2$ for odd n and $g(W) = (n - \text{card}(S) - 1)/2$ for even $n \geq 2$.

We call W *planar* if $g(W) = 0$, that is if W is realized by a coherent curve on S^2 .

Corollary 8.5.5. *An odd full word $W = (w, S \subset \hat{n})$ is planar if and only if $W_{k,l} = 0$ for all $k, l \in \hat{n}$,*

In particular, if W is planar, then $d_S(k)d_S(l) = 0$ for any distinct w -equivalent $k, l \in \hat{n}$. This means that each letter of the alphabet appears in W with at most one superscript $+$. As an exercise, the reader may verify that $g(A^+A^+A) = 1$ and draw coherent curves on \mathbb{R}^2 representing the words $AAAA^+, ABA^+, BAA^+$.

8.6. Example. Pick relatively prime positive integers p, n with $p \leq n$. The associated *Christoffel word* in the 1-letter alphabet $\{A\}$ is the word $W = (w : \hat{n} \rightarrow \{A\}, S)$ where $S = \{i \in \hat{n} \mid [ip/n] = [(i-1)p/n] + 1\}$. Clearly, $\text{card}(S) = p$ so that W is odd iff p is odd. If W is odd, then $g(W) = (n-p)/2$ for odd n and $g(W) = (n-p-1)/2$ for even n .

8.7. Remarks. 1. Deep geometric objects are hiding behind odd words. As we now know, an odd full word W in an alphabet E determines a coherent (pointed) filling curve $f = f(W)$ on a closed oriented surface Σ of genus $g = g(W)$. This gives rise to an algebraic curve over $\overline{\mathbb{Q}}$ and to a point of the moduli space $\mathcal{M}_{g,k}$ where $k = \text{card}(E)$, cf. Introduction and Section 3.7. The curve f also gives rise to an oriented knot in the total space of the tangent circle bundle of Σ . These constructions do not use the base point of f and therefore the resulting geometric objects are preserved under circular permutations of W .

There is a construction of a knot from $f = f(W)$ using its base point. The part of f lying outside of a small open disc $D \subset \Sigma$ surrounding this point is a proper immersed interval on $\Sigma - D$. Proceeding as in [AC], we can derive from this immersed interval a knot in a connected sum of $2g$ copies of $S^1 \times S^2$.

2. Under the correspondence of Corollary 8.5.2, the beaming coherent filling curves correspond to odd full words (w, S) such that S meets each w -equivalence class in one element. The perfect curves correspond to odd full words (w, S) such that S meets each w -equivalence class in its maximal element.

3. A *Gauss word* is a word $W = (w : \hat{n} \rightarrow E, S)$ such that for all $e \in E$, the set $w^{-1}(e)$ has 2 elements and precisely one of them, denoted e^- , belongs to S . Such W is odd and corresponds to a generic curve with no corners. The word W induces a partition of E into two disjoint (possibly empty) subsets: letters $e_1, e_2 \in E$ belong to the same subset iff $e_1^- - e_2^- = 0 \pmod{2}$. One can show that whether W is planar or not depends only on w and this partition of E . Modulo this observation, Corollary 8.5.5 for Gauss words is equivalent to a classical theorem of Rosenstiehl [Ro], [RR], see also [CE].

4. Any (non-signed) full word $W = (w, S = \hat{n})$ can be realized by a filling curve f with $\text{Aut}(f) = \text{Aut}(W)$. Indeed, define a chart $t : \overline{n} \rightarrow \overline{n}$ by $t(k) = -k, t(-k) = v_w(k)$ for $k \in \hat{n}$. Then $\text{Aut}(t) = \text{Aut}(W)$.

9. THE GENUS

9.1. Genus of a chart. Recall that each closed connected (oriented) surface Σ is obtained from the 2-sphere S^2 by attaching several 1-handles. The number of these 1-handles is the genus of Σ . We define the *genus* $g(t)$ of a chart (n, t) to be the genus of a surface containing a filling curve with this chart. The trivial chart has genus 0. Suppose that $n \geq 1$. To compute $g(t)$, recall the circular permutation $\sigma = \sigma_n : \overline{n} \rightarrow \overline{n}$. Define a map $\theta : \overline{n} \rightarrow \overline{n}$ by $\theta(\pm k) = \mp \sigma^{\pm 1}(k)$ for $k \in \hat{n} = \{1, 2, \dots, n\}$. It is easy to check that $\theta^2 = \text{id}$.

Theorem 9.1.1. $g(t) = 1 + (n - \text{card}(\overline{n}/t) - \text{card}(\overline{n}/t\theta))/2$.

Proof. Let $f : S^1 \rightarrow \Sigma$ be a filling curve with chart (n, t) . This curve gives rise to a CW-decomposition X of Σ . Its 0-cells are the points of $\text{cr}(f)$, its 1-cells are the components of $f(S^1) - \text{cr}(f)$, and its 2-cells are the components of $\Sigma - f(S^1)$. Let α_i be the number of i -cells of X . Clearly, $\alpha_0 = \text{card}(\overline{n}/t)$ and $\alpha_1 = n(f) = n$. We claim that $\alpha_2 = \text{card}(\overline{n}/t\theta)$. To see this, recall the flag rotation $t : \text{Fl}(f) \rightarrow \text{Fl}(f)$ and define an involution $\Theta : \text{Fl}(f) \rightarrow \text{Fl}(f)$ as follows. Let $x \in \text{sing}(f) \subset S^1$ and $\varepsilon = \pm$. Starting at x , move along S^1 counterclockwise if $\varepsilon = +$ and clockwise if $\varepsilon = -$. Let y be the first encountered point of $\text{sing}(f)$. Then $\Theta(x, \varepsilon) = (y, -\varepsilon)$. Observe that every flag $r \in \text{Fl}(f)$ gives rise to a 2-cell $D(r)$ of X such that rotating r around its root in the negative direction until meeting $t^{-1}(r) \in \text{Fl}(f)$ we sweep a subarea of $D(r)$. It is easy to see that $D(t\Theta(r)) = D(r)$. Moreover, two flags $r_1, r_2 \in \text{Fl}(f)$ verify $D(r_1) = D(r_2)$ if and only if there is a power of $t\Theta : \text{Fl}(f) \rightarrow \text{Fl}(f)$ transforming r_1 into r_2 . Thus there is a bijective correspondence between the 2-cells of X and the orbits of $t\Theta$. Under the identification $\text{Fl}(f) = \overline{n}$ the involution Θ is transformed into θ . Therefore $\alpha_2 = \text{card}(\overline{n}/t\theta)$. Substituting the expressions for $\alpha_0, \alpha_1, \alpha_2$ in the formula for the Euler characteristic $\alpha_0 - \alpha_1 + \alpha_2 = 2 - 2g(t)$, we obtain the desired formula for $g(t)$. \square

Corollary 9.1.2. For any chart (n, t) , we have $\text{card}(\overline{n}/t) + \text{card}(\overline{n}/t\theta) \leq n + 2$. This inequality becomes an equality if and only if (n, t) is the chart of a curve on S^2 .

Theorem 9.1.1 allows us to compute the genus of specific charts. For the charts $t_1, t_2, t_3 : \overline{4} \rightarrow \overline{4}$ from Section 4.1, $g(t_1) = g(t_3) = 1$ and $g(t_2) = 0$. For the straight chart $t : \overline{4} \rightarrow \overline{4}$ from Section 5.2, $g(t) = 1$.

9.2. Homological formula for the genus. Let (n, t) be a chart with $n \geq 1$. We compute its genus $g(t)$ in homological terms. We shall use notation introduced in the proof of Lemma 3.4.1: the points $x_k = \exp(2k\pi i/n) \in S^1$ with $k \in \hat{n}$, the graph Γ , and the projection $p : S^1 \rightarrow \Gamma$. We first specify a set of generators $\{h_k\}_{k \in \hat{n}}$ for the group $H_1(\Gamma) = H_1(\Gamma; \mathbb{Z})$. Pick $k \in \hat{n}$. The point $v = p(x_k)$ is a vertex of Γ . Consider the path $\gamma_k : [0, 1] \rightarrow S^1$ starting at x_k and moving along S^1 counterclockwise until hitting a point of $p^{-1}(v)$ for the first time. (If $p^{-1}(v) = x_k$, then γ_k traverses the whole circle.) The loop $p\gamma_k : [0, 1] \rightarrow \Gamma$ represents a homology class $h_k \in H_1(\Gamma)$. An induction on the number of vertices of Γ shows that $H_1(\Gamma)$ is generated by $\{h_k\}_{k \in \hat{n}}$. (It is easy to describe generating relations, but we do not need this.)

Let Σ be the closed surface of genus $g = g(t)$ constructed in Lemma 3.4.1 by thickening Γ and gluing 2-disks. The inclusion homomorphism $H_1(\Gamma) \rightarrow H_1(\Sigma)$ is surjective so that the set $\{h_k\}_{k \in \hat{n}}$ generates $H_1(\Sigma) = \mathbb{Z}^{2g}$. The orientation of Σ determines a unimodular bilinear intersection form $B : H_1(\Sigma) \times H_1(\Sigma) \rightarrow \mathbb{Z}$. Hence

$$g(t) = (1/2) \text{rank} B = (1/2) \text{rank}(B(h_k, h_l))_{k, l \in \hat{n}}.$$

In particular, (n, t) is the chart of a curve on S^2 if and only if $B(h_k, h_l) = 0$ for all k, l .

We give now an explicit formula for the integer $B(h_k, h_l)$. We begin with notation. Recall the cyclic order induced by $t : \bar{n} \rightarrow \bar{n}$ on any its orbit (see Section 3.1). For any $r_1, r_2, r_3 \in \bar{n}$, we define an integer $\delta(r_1, r_2, r_3)$: if r_1, r_2, r_3 are pairwise distinct and lie in the same orbit of t in the $(t$ -induced) cyclic order r_1, r_2, r_3, r_1 , then $\delta(r_1, r_2, r_3) = 1$; otherwise $\delta(r_1, r_2, r_3) = 0$. Clearly, $\delta(r_1, r_2, r_3) = \delta(r_2, r_3, r_1) = \delta(r_3, r_1, r_2)$. For $r_1, r_2, r_3, r_4 \in \bar{n}$, set

$$\langle r_1, r_2, r_3, r_4 \rangle = \delta(r_1, r_3, r_2) \delta(r_2, r_4, r_1) - \delta(r_1, r_4, r_2) \delta(r_2, r_3, r_1) \in \mathbb{Z}.$$

Clearly, $\langle r_1, r_2, r_3, r_4 \rangle = 1$ (resp. $\langle r_1, r_2, r_3, r_4 \rangle = -1$) iff r_1, r_2, r_3, r_4 are pairwise distinct and lie in the same t -orbit in the cyclic order r_1, r_3, r_2, r_4, r_1 (resp. r_1, r_4, r_2, r_3, r_1). Otherwise $\langle r_1, r_2, r_3, r_4 \rangle = 0$. We have

$$(9.2.1) \quad \begin{aligned} \langle r_1, r_2, r_3, r_4 \rangle &= -\langle r_3, r_4, r_1, r_2 \rangle, \\ \langle r_1, r_2, r_3, r_4 \rangle &= -\langle r_2, r_1, r_3, r_4 \rangle = -\langle r_1, r_2, r_4, r_3 \rangle. \end{aligned}$$

For $k \in \hat{n}$, let k^+ be the minimal element of the set $\{k+1, k+2, \dots, n\}$ belonging to the orbit of t containing k . If the set $\{k+1, k+2, \dots, n\}$ does not meet the t -orbit of k , then k^+ is the minimal element of the set $\{1, 2, \dots, k\}$ belonging to this orbit. We shall use the cyclic order \prec on \hat{n} and the notation $|k, l|$ introduced in Section 8.5. For $k, l \in \hat{n}$, set

$$\Delta_{k,l} = \begin{cases} 0, & \text{if } k = l, \\ \langle -k^+, k, -l^+, l \rangle, & \text{if } k \prec k^+ \preceq l \prec l^+ \preceq k, \\ -\langle -l^+, l, -l, l^+ \rangle, & \text{if } k \prec l \prec l^+ \prec k^+, \\ \langle -k^+, k, -k, k^+ \rangle, & \text{if } l \prec k \prec k^+ \prec l^+, \\ \langle -k^+, k, -k, k^+ \rangle - \langle -l^+, l, -l, l^+ \rangle, & \text{if } k \prec l^+ \preceq l \prec k^+ \preceq k, \\ \delta(l, -l, -l^+) - \delta(k, -k^+, k^+), & \text{if } k \prec l \prec k^+ \prec l^+ \prec k, \\ \delta(l, -l^+, l^+) - \delta(k, -k, -k^+), & \text{if } l \prec k \prec l^+ \prec k^+ \prec l. \end{cases}$$

Note that $k, k^+, l, l^+ \in \hat{n}$ satisfy three conditions: $k = l$ iff $k^+ = l^+$; if $l = k^+$, then $k^+ \preceq l^+ \preceq k$; if $k = l^+$, then $l^+ \preceq k^+ \preceq l$. The cases listed in the definition of $\Delta_{k,l}$ cover all possibilities for such k, k^+, l, l^+ .

Lemma 9.2.1. *For any $k, l \in \hat{n}$,*

$$(9.2.2) \quad B(h_k, h_l) = \sum_{q \in |k, k^+|, r \in |l, l^+|} \langle -q, q, -r, r \rangle + \sum_{r \in |l, l^+|} \langle -k^+, k, -r, r \rangle + \sum_{q \in |k, k^+|} \langle -q, q, -l^+, l \rangle + \Delta_{k,l}.$$

Proof. To compute the intersection number $B(h, h')$ of homology classes $h, h' \in H_1(\Sigma)$, one presents them by transversal loops $f, f' : S^1 \rightarrow \Sigma$ such that the set of common points $f(S^1) \cap f'(S^1)$ consists only of points of multiplicity 1 on both f and f' . Then one assigns to each common point $+1$ if f crosses f' from left to right at this point and -1 if f crosses f' from right to left. Then $B(h, h') = f \cdot f'$ is sum of these ± 1 's.

Recall the path $\gamma_k : [0, 1] \rightarrow S^1$ defined in Section 9.2. The image of γ_k is an arc on S^1 with endpoints $\gamma_k(0) = x_k$ and $\gamma_k(1) = x_{k^+}$. We denote this arc by the same symbol γ_k and denote the loop $p|_{\gamma_k} : \gamma_k \rightarrow \Gamma \subset \Sigma$ by f_k . By definition, $h_k = [f_k] \in H_1(\Sigma)$ where the square brackets denote the homology class of a loop.

We prove Formula 9.2.2 case by case. Throughout the proof we denote the first, the second, and the third summands on the right-hand side of (9.2.2) by $(I)_{k,l}$, $(II)_{k,l}$, $(III)_{k,l}$ or shorter by (I) , (II) , (III) .

(i) If $k = l$, then $B(h_k, h_l) = \Delta_{k,l} = 0$. We must show that $(I) + (II) + (III) = 0$. By Formula 9.2.1, $\langle -q, q, -q, q \rangle = 0$ and $\langle -q, q, -r, r \rangle + \langle -r, r, -q, q \rangle = 0$ for $q, r \in \hat{n}$. Hence $(I) = (II) + (III) = 0$.

(ii) Suppose that $k \prec k^+ \preceq l \prec l^+ \preceq k$. Then γ_k, γ_l do not meet except possibly at the endpoints. The loops f_k, f_l have a finite set of common points. We analyse separately 4 possible types of common points.

Each pair $q \in |k, k^+|, r \in |l, l^+|$ with $p(x_q) = p(x_r)$ gives a common point of f_k and f_l . The branch of f_k at x_q is formed by two small arcs representing the incoming flag $(x_q, -)$ and the outgoing flag $(x_q, +)$. Under the identification $\text{Fl}(f) = \bar{n}$, these flags correspond to $-q$ and q , respectively. Pick a neighborhood V of $p(x_q) = p(x_r)$ not containing other vertices of Γ . If $\langle -q, q, -r, r \rangle = 0$, then a small deformation of f_k in V makes f_k and f_l disjoint in V . If $\langle -q, q, -r, r \rangle = \pm 1$, then after a small deformation of f_k in V this loop meets f_l transversally in one point whose sign is $\langle -q, q, -r, r \rangle$. Note also that if $q \in |k, k^+|, r \in |l, l^+|$ and $p(x_q) \neq p(x_r)$, then q, r do not lie in the same orbit of t and $\langle -q, q, -r, r \rangle = 0$. Thus the total contribution to $B(h_k, h_l)$ of the pairs $q \in |k, k^+|, r \in |l, l^+|$ with $p(x_q) = p(x_r)$ is equal to (I) .

Each $r \in |l, l^+|$ with $p(x_k) = p(x_r)$ gives a common point of f_k and f_l . The branch of f_k at x_k is formed by two arcs representing the flags $(x_{k^+}, -)$ and $(x_k, +)$. Under the identification $\text{Fl}(f) = \bar{n}$, they correspond to $-k^+$ and k . As above, the contribution of this point to $B(h_k, h_l)$ is $\langle -k^+, k, -r, r \rangle$. Similarly, each $q \in |k, k^+|$ with $p(x_q) = p(x_l)$ contributes $\langle -q, q, -l^+, l \rangle$. This gives (II) and (III). Finally, in the case where $p(x_k) = p(x_l)$, this point contributes $\langle -k^+, k, -l^+, l \rangle = \Delta_{k,l}$.

(iii) Suppose that $k \prec l \prec l^+ \prec k^+$. The points x_l, x_{l^+} split γ_k into three subarcs: an arc α leading from x_k to x_l , the arc γ_l leading from x_l to x_{l^+} and an arc β leading from x_{l^+} to x_{k^+} . Restricting p to α and β we obtain two composable paths in Σ whose composition is a loop, f . It is clear that $h_k = h_l + [f]$. Since B is skew-symmetric, $B(h_k, h_l) = B([f], h_l)$. Since the loops f and f_l meet only at a finite set of points, the same argument as in (ii) computes $B(h_k, h_l) = B([f], h_l)$ to be

$$\sum_{q \in |k, l| \cup |l^+, k^+|, r \in |l, l^+|} \langle -q, q, -r, r \rangle + \sum_{r \in |l, l^+|} \langle -k^+, k, -r, r \rangle + \sum_{q \in |k, l| \cup |l^+, k^+|} \langle -q, q, -l^+, l \rangle + \langle -l, l^+, -l^+, l \rangle.$$

Here $p(x_k) \neq p(x_l)$ and $p(x_r) \neq p(x_l)$ for all $r \in |l, l^+|$ because the set $|k, k^+|$ does not meet the t -orbit of k and the set $|l, l^+|$ does not meet the t -orbit of l . Denote the three big sums in this expression for $B(h_k, h_l)$ by $(I)', (II)', (III)'$, respectively. It is clear that $(II)' = (II)$ and

$$(I) = (I)' + \sum_{q \in |l, l^+|, r \in |l, l^+|} \langle -q, q, -r, r \rangle + \sum_{q \in \{l, l^+\}, r \in |l, l^+|} \langle -q, q, -r, r \rangle = (I)' + 0 + 0 = (I)'.$$

Similarly, $(III)' = (III)$. It remains to observe that $\langle -l, l^+, -l^+, l \rangle = -\langle -l^+, l, -l, l^+ \rangle = \Delta_{k,l}$.

(iv) If $l \prec k \prec k^+ \prec l^+$, then by (iii),

$$B(h_k, h_l) = -B(h_l, h_k) = -(I)_{l,k} - (II)_{l,k} - (III)_{l,k} - \Delta_{l,k}.$$

Formula 9.2.1 implies that $-(I)_{l,k} = (I)_{k,l}$, $-(II)_{l,k} = (II)_{k,l}$, $-(III)_{l,k} = (III)_{k,l}$, and $-\Delta_{l,k} = \Delta_{k,l}$.

(v). Suppose that $k \prec l^+ \preceq l \prec k^+ \preceq k$. If $k = k^+ \neq l = l^+$, then $h_k = h_l = [p]$ and $B(h_k, h_l) = 0$. The right hand side of (9.2.2) is easily computed to be $\sum_{q, r \in \hat{n}} \langle -q, q, -r, r \rangle = 0$ (one should use that $\langle -k, k, -l, l \rangle = 0$ which is due to the fact that $k = k^+$ is the only element in its t -orbit).

Assume that $k \prec l^+ \prec l \prec k^+ \prec k$. (We leave the cases $k \prec l^+ = l \prec k^+ \prec k$ and $k \prec l^+ \prec l \prec k^+ = k$ to the reader.) The circle S^1 splits into four arcs $\alpha, \beta, \alpha', \gamma$ leading from x_k to x_{l^+} , from x_{l^+} to x_l , from x_l to x_{k^+} , and from x_{k^+} to x_k , respectively. Restricting $p : S^1 \rightarrow \Gamma \subset \Sigma$ to α and α' we obtain two composable paths whose composition is a loop, f . Restricting p to β (resp. γ) we obtain a loop g (resp. h) in Σ . Clearly, $h_k = [f] + [g]$ and $h_l = [f] + [h]$. Therefore $B(h_k, h_l) = B(h_k, [h]) + B([g], [f])$. The arcs $\gamma_k = \alpha \cup \beta \cup \alpha'$ and γ have the same endpoints and are otherwise disjoint. The same argument as in (ii) gives

$$(9.2.3) \quad B(h_k, [h]) = \sum_{q \in |k, k^+|, r \in |k^+, k|} \langle -q, q, -r, r \rangle + \sum_{r \in |k^+, k|} \langle -k^+, k, -r, r \rangle + \langle -k^+, k, -k, k^+ \rangle$$

where we use that the set $|k, k^+|$ does not meet the t -orbit of k . The same method gives

$$(9.2.4) \quad B([g], [f]) = \sum_{q \in |l^+, l|, r \in |k, l^+| \cup |l, k^+|} \langle -q, q, -r, r \rangle + \sum_{q \in |l^+, l|} \langle -q, q, -l^+, l \rangle + \langle -l, l^+, -l^+, l \rangle.$$

The absence of further summands is due to the fact that the set $|l^+, l| \subset |k, k^+|$ does not meet the t -orbit of k and the set $|k, l^+| \cup |l, k^+| \subset |l, l^+|$ does not meet the t -orbit of l . Denote the first and the second summands on the right hand side of (9.2.3) (resp. (9.2.4)) by $(I)'$ and $(II)'$ (resp. $(I)''$ and $(II)''$). Observe

that $(II)' = (II)$ since the set $|l, l^+| - |k^+, k|$ meets the t -orbit of k in two points $r = k, k^+$ and for both $\langle -k^+, k, -r, r \rangle = 0$. Also $(II)'' = (III)$ since the set $|k, k^+| - |l^+, l|$ meets the t -orbit of l in two points $q = l, l^+$ and for both $\langle -q, q, -l^+, l \rangle = 0$. These computations yield $B(h_k, h_l) = (I)' + (II)' + (II) + (III) + \Delta_{k,l}$. We claim that $(I)' + (I)'' = (I)$. The equality $|l, l^+| = |l, k^+| \cup |k^+, k| \cup |k, l^+| \cup \{k, k^+\}$ implies that

$$(I) = (I)' + \sum_{q \in |k, k^+|, r \in |k, l^+| \cup |l, k^+|} \langle -q, q, -r, r \rangle.$$

Using that $|k, k^+| = |k, l^+| \cup |l^+, l| \cup |l, k^+| \cup \{l, l^+\}$, we further expand

$$(I) = (I)' + (I)'' + \sum_{q, r \in |k, l^+| \cup |l, k^+|} \langle -q, q, -r, r \rangle + \sum_{q \in \{l, l^+\}, r \in |k, l^+| \cup |l, k^+|} \langle -q, q, -r, r \rangle = (I)' + (I)''$$

since both sums in the middle term are 0.

(vi). Suppose that $k \prec l \prec k^+ \prec l^+ \prec k$. Then $p(x_k) \neq p(x_l)$ and both loops f_k, f_l representing h_k, h_l contain the image of the arc on S^1 leading from x_l to x_{k^+} . For $j \in \bar{n}$, denote a small arc on Γ representing the corresponding flag by α_j . Pushing f_k slightly to its right in Σ , we obtain a loop, f_k^+ , transversal to f_l . We do it so that the point $p(x_k)$ is pushed to a point in $\Sigma - \Gamma$ lying between α_k and $\alpha_{t^{-1}(k)}$. Then $B(h_k, h_l) = f_k^+ \cdot f_l$. In a small neighborhood V of $p(x_l)$ the branch of f_l at x_l is formed by the incoming arc α_{-l^+} and the outgoing arc α_l . The loop f_k goes through $p(x_l)$ as many times as there are elements $q \in |k, k^+|$ lying in the t -orbit of l . (All such q lie in $|k, l| \cup \{l\}$.) For $q \neq l$, the corresponding contribution to $f_k^+ \cdot f_l$ is $\langle -q, q, -l^+, l \rangle$. If α_{-l^+} lies on the left of $\alpha_{-l} \cup \alpha_l$, then the branch of f_k^+ at x_l does not meet the branch of f_l at x_l . If α_{-l^+} lies on the right of $\alpha_{-l} \cup \alpha_l$, then the branch of f_k^+ at x_l meets the branch of f_l at x_l transversally in 1 point, whose sign is +1. This gives a total of

$$\sum_{q \in |k, k^+| - \{l\}} \langle -q, q, -l^+, l \rangle + \delta(l, -l, -l^+) = \sum_{q \in |k, k^+|} \langle -q, q, -l^+, l \rangle + \delta(l, -l, -l^+) = (III) + \delta(l, -l, -l^+).$$

Similarly, the common points of f_k^+, f_l lying a small neighborhood V' of $p(x_k)$ contribute

$$\sum_{r \in |l, l^+|} \langle -k^+, k, -r, r \rangle - \delta(k, -k^+, k^+) = (II) - \delta(k, -k^+, k^+).$$

Finally, the contribution to $f_k^+ \cdot f_l$ of the common points of f_k^+ and f_l lying outside of $V \cup V'$ is (I) .

(vii). The case $l \prec k \prec l^+ \prec k^+ \prec l$ is deduced from the previous one by the same argument as in (iv). \square

9.3. Proof of Theorem 8.5.4. Since the form B in Section 9.2 is unimodular, its rank remains the same after tensor multiplication by $\mathbb{Z}/2\mathbb{Z}$. Therefore to prove Theorem 8.5.4 it suffices to show that $W_{k,l} = B(h_k, h_l) \pmod{2}$ for all $k, l \in \hat{n}$ provided (n, t) is the chart of the coherent curve determined by W . This is a consequence of the following equalities modulo 2. For all $k, l, q, r \in \hat{n}$, we have $\langle -q, q, -r, r \rangle = \langle q, r \rangle$. We have $\langle -k^+, k, -r, r \rangle = d_S(k) \langle k, r \rangle$ if $r \neq k^+$ and $\langle -k^+, k, -r, r \rangle = 0$ if $r = k^+$. Similarly, $\langle -q, q, -l^+, l \rangle = d_S(l) \langle q, l \rangle$ if $q \neq l^+$ and $\langle -q, q, -l^+, l \rangle = 0$ if $q = l^+$. Finally, $\Delta_{k,l} = \Delta_{k,l}^0 \pmod{2}$ because $\langle -k^+, k, -l^+, l \rangle = \langle k, l \rangle d_S(k) d_S(l)$, $\langle -k^+, k, -k, k^+ \rangle = 0$, $\delta(k, -k, -k^+) = d_S(k)$ and $\delta(k, -k^+, k^+) = 1 + d_S(k)$.

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